Mini-course 3 Categorical trace construction

<u>Recall</u> $k=\mathbb{F}_p$, $\mathcal{O}=\mathbb{F}_p[[\varpi]]$ or $\mathbb{Z}_p$. G reductive group $/\mathcal{O}$

$$\rightsquigarrow \mathrm{Gr} = LG/L^+G = \{\mathcal{E} \dashrightarrow \underset{\uparrow \text{trivial } G\text{-torsor}}{\mathcal{E}_{triv}} \text{ modification of } G\text{-torsors}/\mathrm{Spec}\,\mathcal{O}\}$$

$$L^+G \curvearrowright \; = \coprod_{\lambda\in X_*(T)^+} \underbrace{L^+G\varpi^\lambda L^+G/L^+G}_{\overset{\parallel}{\mathring{\mathrm{Gr}}_\lambda}} = \{\mathcal{E} \overset{\lambda}{\dashrightarrow} \mathcal{E}_{triv}\}$$

Put $\mathrm{Hk} := [L^+G\backslash \mathrm{Gr}] = \{\mathcal{E} \dashrightarrow \mathcal{E}' \text{ modification of } G\text{-torsors}/\mathrm{Spec}\,\mathcal{O}\}$

Can define "$\mathrm{Perv}(\mathrm{Hk}) := \mathrm{Perv}_{L^+G}(\mathrm{Gr})$" by truncation

(Given $\mu\in X_*(T)$, L^+G action on Gr_μ factors through $L^+G = G(\mathcal{O}) \twoheadrightarrow L^NG = G(\mathcal{O}/\varpi^N)$

Define $\mathrm{Perv}_{L^+G}(\mathrm{Gr}) = \varinjlim_{\underbrace{\mu\in X_*(T)^+}_{\uparrow \text{assuming } \pi_0(LG)=\pi_1(G)=\{1\}}} \varprojlim_N \mathrm{Perv}_{L^NG}(\mathrm{Gr}_\mu)$)

<u>Theorem</u> ("Absolute" geometric Satake)

There is an equivalence of tensor categories

$$\mathrm{Perv}(\mathrm{Hk}) \simeq \mathrm{Perv}_{L^+G}(\mathrm{Gr}) \xrightarrow[H^*(\mathrm{Gr},-)]{} \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$$

$$j_{\mu!*}\bar{\mathbb{Q}}_\ell[\langle 2\rho,\mu\rangle] = IC_{\mathrm{Gr}_\mu} \longleftrightarrow V_\mu$$

$$\text{Convolution product } \mathcal{F} * \mathcal{G} \longleftrightarrow V\otimes W.$$

<u>Geometric Satake vs. classical Satake</u>

$$P^0_{L^+G}(\mathrm{Gr}) = \{L^+G\text{-equivariant perverse sheaves, pure of weight } 0\}$$

$\cup$ simple objects: $\underset{\text{if } \sigma(\mu)=\mu,}{IC^N_\mu}$ s.t. $IC^N_\mu|_{\mathring{\mathrm{Gr}}_\mu} = \bar{\mathbb{Q}}_\ell[\langle 2\rho,\mu\rangle](\langle\rho,\mu\rangle)$ ← choice of $\bar{\mathbb{Q}}_\ell(\tfrac{1}{2})$

otherwise, take $\bigoplus_{\mu'\in\langle\sigma\rangle\mu} IC^N_{\mu'}$.

Then Satake equivalence upgrades to

$$\mathcal{P}^{\circ}_{L^+G}(\mathrm{Gr}) \xrightarrow{\sim} \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LG)$$

<u>Theorem</u> The following diagram commutes (ϕ for geometric Frobenius)

$$\begin{tikzcd}
{[\mathcal{F}] \in K_0(\mathcal{P}^{\circ}_{L^+G}(\mathrm{Gr}))} \arrow[r, "\mathrm{Sat}"] \arrow[d, "\text{Sheaf-function dictionary}"] & K_0(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LG)) \arrow[r] & \bar{\mathbb{Q}}_\ell[\hat{G}\phi/\mathrm{Ad}\hat{G}] \simeq \bar{\mathbb{Q}}_\ell[\hat{T}\phi/\hat{T}]^W \arrow[d, equal] \\
\mathrm{Hk}_G = C_c^\infty(\underbrace{G(\mathcal{O})\backslash G(E)/G(\mathcal{O})}_{\mathrm{Gr}(k)}, \bar{\mathbb{Q}}_\ell) \arrow[r] & C_c^\infty(T(\mathcal{O})\backslash T(E), \bar{\mathbb{Q}}_\ell)^{W_0} = \bar{\mathbb{Q}}_\ell[X_*(T)^\phi]^{W_0} & \bar{\mathbb{Q}}_\ell[X^*(\hat{T})^\phi]^W \arrow[l, equal]
\end{tikzcd}$$

$[\mathcal{F}] \mapsto (x \mapsto \mathrm{Tr}(\phi, [H^\cdot(\mathcal{F}_x)]))$

$[V] \longmapsto \chi_V|_{G\phi}$

<u>A subtle remark</u>: There are two Satake isomorphisms ϕ for geom. Frob, $\sigma = \phi^{-1}$ for arithmetic

$$\begin{tikzcd}
 & \bar{\mathbb{Q}}_\ell[\hat{G}\phi/\mathrm{Ad}\hat{G}] \arrow[r, "\simeq"] \arrow[d, "g\phi \mapsto (g\phi)^{-1} = \sigma(g^{-1})\sigma"] & \bar{\mathbb{Q}}_\ell[X^\cdot(\hat{T})^\phi]^W \arrow[d, equal, "\lambda \mapsto -\lambda"] \\
K_0(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^LG)) \arrow[ur] \arrow[dr] \arrow[urr, bend left, "\mathrm{Sat}^{cl}"] \arrow[drr, bend right, "\mathrm{Sat}^{cl'}"'] & & \\
 & \bar{\mathbb{Q}}_\ell[\hat{G}\sigma/\mathrm{Ad}\hat{G}] \arrow[r, "\simeq"] & \bar{\mathbb{Q}}_\ell[X^\cdot(\hat{T})^\sigma]^W
\end{tikzcd}$$

will use this later

$$\mathrm{Sat}^{cl}(V) = \mathrm{Sat}^{cl'}(V^*)$$

omit

§ Local Shtukas

<u>Definition</u> Given dominant coweights $\mu_1, \cdots, \mu_r \in X_*(T)^+$ $\mu_\bullet := (\mu_1, \cdots, \mu_r)$

$$\mathrm{Hk}^{loc}_{\mu_\bullet} := \left\{ \mathcal{E}_r \overset{\le \mu_r}{\dashrightarrow} \mathcal{E}_{r-1} \overset{\le \mu_{r-1}}{\dashrightarrow} \cdots \dashrightarrow \mathcal{E}_1 \overset{\le \mu_1}{\dashrightarrow} \mathcal{E}_0 \right\}$$

For a (perfect) k-algebra R and $\mathcal{E}$ a G-torsor over $\mathrm{Spec}\, R[\![\varpi]\!]$ or $\mathrm{Spec}\, W(R)$.

define ${}^\sigma\mathcal{E} := (\sigma_R \otimes \mathrm{id})^* \mathcal{E}$.

Define the prestack $\mathrm{Sht}^{loc}_{\mu_\bullet}$ to classify

- an R-point of $Hk^{loc}_{\mu_\bullet}$: $\mathcal{E}_r \overset{\leq \mu_r}{\dashrightarrow} \cdots \overset{\leq \mu_1}{\dashrightarrow} \mathcal{E}_0$
- and an isomorphism $\mathcal{E}_0 \simeq {}^{\sigma}\mathcal{E}_r$

<u>We have</u>:

$$\begin{tikzcd} Sht^{loc}_{\mu_\bullet} \arrow[r, "\Phi^{loc}"] \arrow[d] & Hk^{loc}_{\mu_\bullet} \arrow[d, "\overleftarrow{h} \times \overrightarrow{h}"] \\ \mathbb{B}L^+G \arrow[r, "1 \times \sigma"] & \mathbb{B}L^+G \times \mathbb{B}L^+G \end{tikzcd}$$

φ forgets the isomorphism $\mathcal{E}_0 \simeq {}^{\sigma}\mathcal{E}_r$.

<u>Fact</u>: Given a Shimura variety $Sh_G(K)_{/\mathbb{F}_{p^N}}$ of Hodge type with K_p hyperspecial

$$\begin{tikzcd} Sh_G(K)^{pf} \arrow[d, "loc_p"] & A \arrow[d, mapsto] \\ Sht^{loc}_{\mu} & \mathcal{D}' \end{tikzcd}$$

gives $\mu : \mathbb{G}_m \to G$ minuscule cocharacter

$\mathcal{D}' := \mathcal{D}(A[p^\infty])$, $V : \mathcal{D}' \xrightarrow{\sigma(\mu)} \sigma(\mathcal{D}')$

Consider $\mathbb{D} := (\sigma^{-1}(\mathcal{D}'), V)$ this defines $loc_p : Sh_G(K)^{pf} \to Sht^{loc}_{G,\mu}$

<u>Hope</u>:

$$\begin{tikzcd} \underset{(A,\lambda,\eta)}{Sh_G(K)} \arrow[d, "loc_p"] \arrow[dr, phantom, "\square"] & \underset{\text{quasi-isog. } A \to B}{I} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underset{(B,\lambda,\eta)}{Sh_{G'}(K)} \arrow[d, "loc_p"] \\ Sht^{loc}_{\mu} & Sht^{loc}_{\mu|\lambda} \arrow[l] \arrow[r] & Sht^{loc}_{\lambda} \end{tikzcd}$$

$G(\mathbb{A}_f) \simeq G'(\mathbb{A}_f)$

$\mu \qquad \lambda$

$\mathcal{E} \xrightarrow{\mu} {}^{\sigma}\mathcal{E}$

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\mu"] \arrow[d, dashed, "\beta"] & {}^{\sigma}\mathcal{E} \arrow[d, dashed, "\sigma(\beta)"] \\ \mathcal{E}' \arrow[r, "\lambda"'] & {}^{\sigma}\mathcal{E}' \end{tikzcd}$$

$\mathcal{E}' \xrightarrow{\lambda} {}^{\sigma}\mathcal{E}'$

$\rightsquigarrow$ Define $Sht^{loc,\nu}_{\mu|\lambda}$ to be such moduli prestack with $\mathrm{inv}(\beta) \leq \nu$.

<u>Definition</u> "space" = schemes, stacks, ind-stacks, $\cdots$ $/k$

Suppose given a correspondence

$$\begin{tikzcd} & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ X_1 & & X_2 \end{tikzcd}$$

and $\mathcal{F}_1 \in D_b(X_1)$, $\mathcal{F}_2 \in D_b(X_2)$

A <u>cohomological correspondence</u> is a map $u : c_1^* \mathcal{F}_1 \to c_2^! \mathcal{F}_2$

$$Corr_C((X_1, \mathcal{F}_1), (X_2, \mathcal{F}_2)) := Hom_C(c_1^* \mathcal{F}_1, c_2^! \mathcal{F}_2)$$

⇝ This gives $H_c^*(X_{1,\bar k}, \mathcal{F}_1) \xrightarrow{c_1^*} H_c^*(C_{\bar k}, c_1^*\mathcal{F}) \xrightarrow{H_c^*(u)} H_c^*(C_{\bar k}, c_2^!\mathcal{F}_2) \xrightarrow{Tr} H_c^*(X_{2\bar k}, \mathcal{F}_2)$

Can define compositions of cohom. corresp.

* pushforward:

$$\begin{tikzcd} X_1 \arrow[d, "f_1"'] & C \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[d, "f"] & X_2 \arrow[d, "f_2"] \\ Y_1 & D \arrow[l, "d_1"'] \arrow[r, "d_2"] & Y_2 \end{tikzcd}$$

"base changable"

$C \to X_1 \times_{Y_1} D$ is rep'd by perfection of proper alg. space

$u : (X_1, \mathcal{F}_1) \longrightarrow (X_2, \mathcal{F}_2)$

$\rightsquigarrow$

$f_!(u) : (Y_1, f_{1!}\mathcal{F}_1) \longrightarrow (Y_2, f_{2!}\mathcal{F}_2)$

* pullback: Assume that f is cohomologically smooth, equi-dim. of rel. dim d.

$v : (Y_1, \mathcal{G}_1) \longrightarrow (Y_2, \mathcal{G}_2)$

$\rightsquigarrow f^*(v) : (X_1, f_1^*\mathcal{G}_1) \longrightarrow (X_2, f_2^!\mathcal{G}_2[-2d_f](-d_f))$

omit.

Idea: Define cohom. corresp. between Sht_μ^{loc}'s & pullback to Shimura varieties

Interested in:

$$\begin{tikzcd} \mathrm{Rep}_{\mathbb{Q}_\ell}(\hat G) \arrow[r, "\sim"] & \mathrm{Perv}(\mathrm{Hk}_{\bar k}) \arrow[d] \\ & \mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{loc}_{\bar k}) \end{tikzcd}$$

$\mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{loc}_{\bar k})$ = category of perverse sheaves on $\mathrm{Sht}^{loc}_{\bar k}$, + morphisms are cohom. corresp.

$$\begin{tikzcd} V_\mu \arrow[r, mapsto] & IC_\mu \arrow[d, mapsto] \\ & \Phi^* IC_\mu \end{tikzcd}$$

Object-wise, there's nothing new.

$$\begin{tikzcd} & \mathrm{Sht}^{loc}_{\mu|\lambda} \arrow[dl] \arrow[dr] & \\ (\mathrm{Sht}^{loc}_\mu, \mathcal{F}_1) & & (\mathrm{Sht}^{loc}_\lambda, \mathcal{F}_2) \end{tikzcd}$$

Slogan. Galois representations determine the cohomological correspondences.

* stack of unramified Langlands parameters

$$\mathrm{Loc}_{{}^LG} : \mathrm{Hom}_{gp}\Big(\underset{\langle\sigma\rangle}{\overset{\shortparallel}{W_{\mathbb{F}_p}}} \longrightarrow \hat G \rtimes \langle\sigma\rangle\Big)\Big/\mathrm{Ad}\,\hat G = \big[\hat G\sigma / \mathrm{Ad}\,\hat G\big] = \big[\hat G / \mathrm{Ad}_\sigma \hat G\big]$$

$$\begin{tikzcd} W_{\mathbb{F}_p} \arrow[r] \arrow[dr, "id"'] & \hat G \rtimes \langle\sigma\rangle \arrow[d] \\ & \langle\sigma\rangle \end{tikzcd}$$

$c_\sigma(g)\cdot h = g h \sigma(g)^{-1}$

Expectation $H_c^*(\mathrm{Sh}_K(G)_{\bar{\mathbb{F}}_p}, \bar{\mathbb{Q}}_\ell) = \Gamma(\mathrm{Loc}_{{}^LG}, \underbrace{\mathcal{A}}_{\text{skyscraper sheaf} = \bigoplus_i \delta_{\rho_i}} \otimes \tilde V_\mu)$

where $[\hat G/\mathrm{Ad}_\sigma \hat G] \xrightarrow{\pi} [pt/\hat G]$

$\tilde V_\mu := \pi^* V_\mu \qquad V_\mu$

$$= \bigoplus_i r_\mu \circ \delta_{\rho_i}$$

Thus, $H^*_c(Sh_K(G)_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \otimes \mathrm{Hom}_{[\hat{G}/\mathrm{Ad}_\sigma \hat{G}]}(\tilde{V}_\mu, \tilde{V}_\lambda) \longrightarrow H^*_c(Sh_K(G')_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$

<u>Better</u>:

$$\underset{\substack{\uparrow \\ \text{full subcat of objs} = \bigoplus \tilde{V}_\mu}}{\mathrm{Coh}_{fr}}\left([\hat{G}/\mathrm{Ad}_\sigma \hat{G}]\right) \xrightarrow{S^?} \mathrm{Perv}^{\mathrm{Corr}}\left(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}\right) \underset{\substack{\uparrow \\ \text{in the minuscule case}}}{\overset{\mathrm{loc}_p^*}{\dashrightarrow}} \mathrm{Perv}^{\mathrm{Corr}}\left(Sh_K(G)\right)$$

<u>Theorem</u> $\exists$ a functor $S : \mathrm{Coh}_{fr}([\hat{G}/\mathrm{Ad}_\sigma \hat{G}]) \longrightarrow \mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}})$ s.t. the following diagram commutes

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Q}_\ell}(\hat{G}) \arrow[r, "\mathrm{Sat}", "\sim"'] \arrow[d, "\pi^*"] & \mathrm{Perv}(\mathrm{Hk}_{\bar{k}}) \arrow[d, "\Phi^{\mathrm{loc},*}"] \\
\mathrm{Coh}_{fr}([\hat{G}/\mathrm{Ad}_\sigma \hat{G}]) \arrow[r, "S"] & \mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}})
\end{tikzcd}$$

* <u>Abstract proof</u> (categorical trace)
    * E = base commutative ring

      A = E-alg, not necessarily commutative

      $\mathrm{Tr}(A) := A/(ab - ba;\ a, b \in A)$ as a quotient of E-modules

      $= \mathrm{colim}\left(A \otimes A \rightrightarrows A\right)$

      Universal property: universal for maps $f : A \longrightarrow V$ $\quad V = E\text{-mod},\ f(ab) = f(ba)$

* In general, $(\mathcal{C}, \otimes, \mathbb{1})$ an E-linear monoidal category

  $\sigma : \mathcal{C} \longrightarrow \mathcal{C}$ automorphism.

  Define the (twisted) categorical trace to be the 2-colimit of

$$\mathcal{C}^{\otimes 3} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{C}^{\otimes 2} \rightrightarrows \mathcal{C} \qquad \rightsquigarrow \text{denoted by } \mathrm{Tr}_\sigma(\mathcal{C})$$

$$(X, Y) \longmapsto \begin{cases} X \otimes Y \\ \sigma Y \otimes X \end{cases}$$

$$(X, Y, Z) \longmapsto \begin{cases} (X, Y \otimes Z) \\ (\sigma Z \otimes X, Y) \\ (Z, X \otimes Y) \end{cases}$$

Universal for
$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D} \\ & \mathrm{Tr}_\sigma(\mathcal{C}) \arrow[ur, dashed] & \end{tikzcd}$$
s.t. $\exists$ functorial $\alpha_{X,Y}: F(X\otimes Y) \simeq F(\sigma Y \otimes X)$

<u>Proposition</u> ① Assume that every object V in $\mathcal{C}$ admits a <u>left dual</u> V^*. Then $\mathrm{Tr}_\sigma(\mathcal{C})$ exists

<u>left dual</u>: $\mathrm{coev}_V: 1 \to V\otimes V^*$, $\mathrm{ev}_V: V^*\otimes V \to \mathbb{1}$ satisfying obvious conditions

② When $\mathcal{C} = \mathrm{DCoh}(X)$ with X smooth and $\sigma: X \xrightarrow{\sim} X$

$$\mathrm{Tr}_\sigma(\mathcal{C}) = \mathrm{DCoh}\Big(X \underset{\Delta \searrow X\times X \swarrow_{1\times\sigma}}{\times} X\Big) = \mathcal{L}_\sigma(X)$$

<u>Main example</u>: $X = [\,\cdot/\hat{G}\,]$,

$$\mathcal{L}_\sigma(X) = \Big[\, \cdot/\hat{G} \underset{\Delta \searrow \cdot/\hat{G}\times \cdot/\hat{G} \swarrow_{1\times\sigma}}{\times} \cdot/\hat{G} \,\Big]$$
moduli of $\hat{G}$-torsors $\mathcal{E}_1$ and $\mathcal{E}_2$ & isoms $\alpha: \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$, $\beta: \mathcal{E}_1 \xrightarrow{\sim} {}^\sigma\mathcal{E}_2$

$=$ moduli of $\hat{G}$-torsor $\beta: \mathcal{E}_2 \xrightarrow{\sim} {}^\sigma\mathcal{E}_2$

$$= [\,\hat{G}/\mathrm{Ad}_\sigma \hat{G}\,]$$

<u>Proof of Proposition $\Rightarrow$ Theorem</u>

Use universal properties:
$$\begin{tikzcd} \mathrm{Coh}([\cdot/\hat{G}]) \arrow[rr, "\mathrm{Sat}", "\simeq"'] \arrow[d] & & \mathrm{Perv}(\mathrm{Hk}_{\bar{k}}) \arrow[d, "\Phi^{\mathrm{loc},*}"] \\ \mathrm{Tr}_\sigma(\mathrm{DCoh}([\cdot/\hat{G}])) = \mathrm{DCoh}([\hat{G}/\mathrm{Ad}_\sigma \hat{G}]) \arrow[rr, dashed, "\text{universal}", "\text{property}"'] & & \mathrm{Perv}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}}_{\bar{k}}) \end{tikzcd}$$

need isom. for IC_μ, IC_λ, $\alpha_{\mu,\lambda}: IC_{\mathrm{Sht}^{\mathrm{loc}}_\mu} \tilde{\otimes} IC_{\mathrm{Sht}^{\mathrm{loc}}_\lambda} \simeq IC_{\mathrm{Sht}^{\mathrm{loc}}_{\sigma(\lambda)}} \tilde{\otimes} IC_{\mathrm{Sht}^{\mathrm{loc}}_\mu}$

$$\alpha_{\mu,\lambda}: (\mathrm{Sht}^{\mathrm{loc}}_{\mu,\lambda}, IC_{\mu,\lambda}) \xrightarrow{\overset{?}{\sim}} (\mathrm{Sht}^{\mathrm{loc}}_{\sigma(\lambda),\mu}, IC_{\sigma(\lambda),\mu})$$

$$\Big(\mathcal{E}_2 \xrightarrow{\lambda} \mathcal{E}_1 \xrightarrow{\mu} {}^\sigma\mathcal{E}_2\Big) \longmapsto \Big(\mathcal{E}_1 \xrightarrow{\mu} {}^\sigma\mathcal{E}_2 \xrightarrow{\sigma(\lambda)} {}^\sigma\mathcal{E}_1\Big)$$

isomorphism after perfection!

<u>Concrete proof</u>:

① Construction of $\mathrm{Tr}_\sigma(\mathcal{C})$:

Objects $V \in \mathcal{C} \rightsquigarrow \tilde{V} \in \mathcal{C}$

Define $\mathrm{Hom}_{Tr_\sigma(\mathcal{C})}(\tilde{X}, \tilde{Y}) := \left(\bigoplus_{(V,W)\in \mathcal{C}\times\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X, V\otimes W) \otimes \mathrm{Hom}_{\mathcal{C}}(\sigma W \otimes V, Y) \right) \Big/ \sim$

where the equivalence is that whenever

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "u'"'] & V\otimes W \arrow[d, "\sum \eta_i\otimes\zeta_i \in \mathrm{Hom}_{\mathcal{C}}(V,V')\otimes \mathrm{Hom}_{\mathcal{C}}(W,W')"] \\ & V'\otimes W' \end{tikzcd} \qquad \begin{tikzcd} \sigma W\otimes V \arrow[r, "v"] \arrow[d, "\sum \sigma\zeta_i\otimes\eta_i"'] & Y \\ \sigma W'\otimes V' \arrow[ur, "v'"'] & \end{tikzcd}$$

we assume that $u\otimes v \sim u'\otimes v'$

② In our special case of $\mathrm{Coh}([\cdot/\hat{G}])$

$$\mathrm{Hom}_{[\hat{G}/Ad_\sigma \hat{G}]}(\tilde{X},\tilde{Y}) = \mathrm{Hom}_{\hat{G}}(\mathcal{O}_{\hat{G}}\otimes X, \mathcal{O}_{\hat{G}}\otimes Y)^{Ad_\sigma \hat{G}} = (\mathcal{O}_{\hat{G}}\otimes X^*\otimes Y)^{Ad_\sigma(\hat{G})}$$

$$\mathrm{Hom}_{\hat{G}}(X, V\otimes W) \otimes \mathrm{Hom}_{\hat{G}}(\sigma W\otimes V, Y) \longrightarrow \big(\underbrace{\mathcal{O}_{\hat{G}}\otimes X^*\otimes Y}_{\substack{\| \\ f:\hat{G}\to \mathrm{Hom}_{\bar{\mathbb{Q}}_\ell}(X,Y)}}\big)^{Ad_\sigma(\hat{G})}$$

$G \xrightarrow{\sigma^{-1}} G \longrightarrow GL(W)$

$f(hg\sigma(h)^{-1}) = h\circ f(g)\circ h^{-1}$

$$(\eta, \zeta) \longmapsto (g \longmapsto \zeta\circ(g\times 1)\circ \mathrm{swap}_\sigma\circ\eta)$$

Note $\zeta\circ(hg\sigma(h)^{-1}\times 1)\circ \mathrm{swap}_\sigma\circ\eta = h\circ\zeta\circ(g\times 1)\mathrm{swap}_\sigma\circ\eta\circ h^{-1}$

need $(\sigma(h)^{-1}\times h^{-1})\circ \mathrm{swap}_\sigma = \mathrm{swap}_\sigma\circ(h^{-1}\times h^{-1})$

③ Geometric realization

$$\begin{tikzcd}[column sep=small] & & & \mathrm{Sht}^{loc,\nu}_{\lambda|\mu} \arrow[dl] \arrow[dr] & & & \\ & & \mathrm{Sht}^{loc,0}_{\lambda|\sigma(\nu^*),\eta} \arrow[dll] \arrow[dl] & & \mathrm{Sht}^{loc,0}_{(\eta,\nu^*)|\mu} \arrow[dl] \arrow[dr] & & \\ \mathrm{Sht}^{loc}_{\lambda} & \mathrm{Sht}_{\sigma(\nu^*),\eta} \arrow[rr, "F^{-1}"] & & \mathrm{Sht}^{loc}_{(\eta,\nu^*)} & & \mathrm{Sht}^{loc}_{\mu} & \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}' \arrow[r, "\lambda"] \arrow[d, "\nu"'] \arrow[dr, dashed, "\eta"] & {}^\sigma\mathcal{E}' \arrow[d, "\sigma(\nu)"] \\ \mathcal{E} \arrow[r, "\mu"'] & {}^\sigma\mathcal{E} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{E}' \arrow[r, dashed, "\eta"'] \arrow[rr, bend left, "\lambda"] & {}^\sigma\mathcal{E} \arrow[r, "\sigma(\nu^*)"'] & {}^\sigma\mathcal{E}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "\nu^*"'] \arrow[rr, bend left, "\mu"] & \mathcal{E}' \arrow[r, dashed, "\eta"'] & {}^\sigma\mathcal{E} \end{tikzcd}$$

$\mathrm{Hk}_{\lambda|\sigma(\nu^*),\eta}$ $\qquad$ $\mathrm{Hk}^{\sigma}_{(\eta,\nu^*)|\mu}$

$\mathrm{Hk}_\lambda \leftarrow \mathrm{Hk}_{\lambda|\sigma(\nu^*),\eta} \rightarrow \mathrm{Hk}_{\sigma(\nu^*),\eta} \qquad \mathrm{Hk}_{(\eta,\nu^*)} \leftarrow \mathrm{Hk}_{(\eta,\nu^*)|\mu} \rightarrow \mathrm{Hk}_\mu$

$\mathcal{E}' \xrightarrow{\lambda} {}^\sigma\mathcal{E}' \qquad \mathcal{E}' \overset{\eta}{\dashrightarrow} {}^\sigma\mathcal{E} \to {}^\sigma\mathcal{E}' \longleftarrow\!\shortmid (\mathcal{E} \to \mathcal{E}' \overset{\eta}{\dashrightarrow} {}^\sigma\mathcal{E}) \qquad \mathcal{E} \to {}^\sigma\mathcal{E}$

Define a morphism $\mathrm{Hom}_{\hat{G}}(V_\mu, V\otimes W) \otimes \mathrm{Hom}_{\hat{G}}(\sigma W \otimes V, V_\lambda)$

$$\downarrow$$

$$\mathrm{Corr}\big((\mathrm{Hk}_\mu, \mathrm{IC}_\mu), (\mathrm{Hk}_{(\eta,\nu^*)}, \mathrm{Sat}(V) * \mathrm{Sat}(W))\big) \otimes \mathrm{Corr}\big((\mathrm{Hk}_{(\sigma(\nu^*),\eta)}, \mathrm{Sat}(\sigma W) * \mathrm{Sat}(V)), (\mathrm{Hk}_\lambda, \mathrm{IC}_\lambda)\big)$$

$$\downarrow \qquad (\phi, \eta) \mapsto (\text{pullback of } \eta) \circ (F^{-1})^* \circ (\text{pullback of } \phi).$$

$$\mathrm{Corr}\big((\mathrm{Sht}_\mu, \mathrm{IC}_\mu), (\mathrm{Sht}_\lambda, \mathrm{IC}_\lambda)\big)$$

<u>Proposition</u>: $\exists!$ a factorization

$$\begin{array}{ccc} \mathrm{Hom}_{\hat{G}}(V_\mu, V\otimes W) \otimes \mathrm{Hom}_{\hat{G}}(\sigma W \otimes V, V_\lambda) & \longrightarrow & \mathrm{Hom}_{[\hat{G}/\mathrm{Ad}_\sigma \hat{G}]}(\widetilde{V}_\mu, \widetilde{V}_\lambda) \\ \downarrow & \swarrow \exists! & \\ \mathrm{Corr}\big((\mathrm{Sht}_\mu, \mathrm{IC}_\mu), (\mathrm{Sht}_\lambda, \mathrm{IC}_\lambda)\big) & & \end{array}$$

<u>What's left to do</u>? Given $f \in \mathrm{Hom}_{[\hat{G}/\mathrm{Ad}_\sigma \hat{G}]}(\widetilde{V}_\mu, \widetilde{V}_\lambda)$

what's the support of $S(f)$? Next time.