# Coherent sheaves on moduli stack of Langlands parameters

## §1. Moduli stack of Galois rep'ns.

Coefficients $\Lambda = \mathbb{F}_\ell, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, $\ell \neq p$.

$F_x$ local field, $F_x \supseteq \mathcal{O}_x \twoheadrightarrow \mathcal{O}_x/\varpi \simeq \mathbb{F}_q$. $q = p^r$.

$F = \mathbb{F}_q(X)$ function field.

$\hat{G}$ affine alg. gp $/\Lambda$.

$D \subseteq X$ finite set of points

$$\begin{array}{ccccccccc} 1 & \to & I_{F_x} & \to & \mathrm{Gal}_{F_x} & \to & \sigma^{\hat{\mathbb{Z}}} & \to & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \to & I_{F_x} & \to & W_{F_x} & \to & \sigma^{\mathbb{Z}} & \to & 1 \\ & & \cup & & & & & & \\ & & P_{F_x} & & & & & & \end{array}$$

$P_{F_x}$ ← pro-$p$ group

$\ni \sigma, \tau \in I_{F_x}/P_{F_x}$

$$\begin{array}{ccccccc} 1 \to \pi_1^{\mathrm{geom}}(X \setminus D) & \to & \pi_1(X \setminus D) & \to & \sigma^{\hat{\mathbb{Z}}} & \to & 1 \\ & & \cup & & \cup & & \\ & & W_{F,D} & \to & \sigma^{\mathbb{Z}} & \to & 1 \end{array}$$

$$\mathrm{Loc}_{W_{F_x}, \hat{G}} := \Big[ \underbrace{\mathrm{Hom}_{\mathrm{cont}}(W_{F_x}, \hat{G})}_{R_{W_{F_x}, \hat{G}}} \Big] \Big/ \hat{G}$$

$\mathrm{Loc}_{W_{F,D}, \hat{G}}$ def'd similarly.

**Remarks** ① Local case : as "$\#P_{F_x}$" is invertible in $\Lambda$,

$$\mathrm{Loc}_{W_{F_x}, \hat{G}} = \coprod_{\rho_0 : P_{F_x} \to \hat{G}(\Lambda)} \mathrm{Loc}_{\Gamma_q, \mathrm{Cent}_{\hat{G}}(\rho_0)}$$

← "blocks": but each block "looks alike"

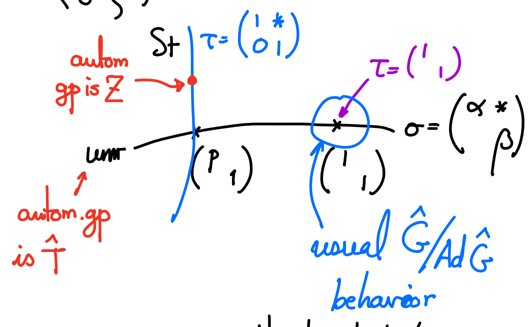
$\infty$-disjoint union

$\Gamma_q = \langle \sigma, \tau \mid \sigma\tau\sigma^{-1} = \tau^q \rangle$ ← note image of $\tau$ is forced to be top. unip.

E.g. $\hat{G} = GL_2$. $\rho_0 = \mathbb{1}$. $R^{\mathrm{tame}}_{W_{\mathbb{Q}_p}, GL_2} = \left\{ (\sigma, \tau) \in GL_2 \;\middle|\; \sigma\tau\sigma^{-1} = \tau^p \right\} / \Lambda$

"Up to conjugation" $\sigma = \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix}$ (usually 0), $\tau = \begin{pmatrix} \zeta & * \\ 0 & \zeta' \end{pmatrix}$ $\zeta^{p-1} = \zeta'^{p-1} = 1$.

$$R^{\mathrm{unip}}_{W_{\mathbb{Q}_p}, GL_2} := \{ \tau \text{ unipotent} \}$$
$$\cup$$
$$R^{\mathrm{unr}}_{W_{\mathbb{Q}_p}, GL_2}$$




Global case: ... ← quasi-smooth alg. stack $/\mathbb{Z}_\ell$.

$$\mathrm{Loc}_{W_{F,D},\hat{G}} = \coprod_{\substack{\theta:\text{ pseudo rep}\\ \text{of } \pi_1^{geom}(X-D)}} \mathrm{Loc}_{W_{F,D},\hat{G}}$$

This is non-trivial! Uses de Jong's conjecture (proved by Gaitsgory)

If $\rho: W_{F,D} \to GL_n(\mathbb{F}_\ell((t)))$ is a continuous rep'n,

then $\rho(\pi_1^{geom}(X-D))$ is finite.

② In general, $R_{W_{F(x)},\hat{G}}$ has derived structure,

but when $F_x$ local + $\hat{G}$ reductive $\Rightarrow R_{W_{F_x},\hat{G}}$ is classical.

③ (Assume $\Lambda$ is a field, $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$)

Say $(\rho: W_{F(x)} \to \hat{G}) \in \mathrm{Loc}_{W_{F(x)},\hat{G}}$ an <u>elliptic point</u> if $\mathrm{Cent}_{\hat{G}}(\rho) \overset{\text{finite}}{\supseteq} Z_{\hat{G}}$

$\rightsquigarrow S_\rho := \mathrm{Cent}_{\hat{G}}(\rho)/Z_{\hat{G}}$

In this case $[*/S_\rho] \hookrightarrow \mathrm{Loc}_{W_{F(x)},\hat{G}}$

open and closed

This $S_\rho$ is related to $\mathcal{L}$-parameters

④ Should have used ${}^cG = \hat{G} \rtimes (\mathbb{G}_m \times \mathrm{Gal}_{F(x)})$ instead

but over $\mathbb{Z}[q^{\pm 1/2}]$, this is the same as ${}^LG$ (or $\hat{G}$ if $G$ splits).

## §2 Local and global conjectures

- $X$ smooth geom. conn. proj. curve $/\mathbb{F}_q$

$\overset{\cup\!\shortmid}{D}$. $\quad I = \{1,\dots,m\}$ finite set, $K$ level set, $K = \prod_{x\notin|D|} G(\mathcal{O}_x) \times \prod_{x\in|D|} K_x$

$W = \boxtimes V_\mu \in \mathrm{Rep}(\hat{G}^I)$

$\rightsquigarrow$ $IC_{\mu_\bullet}$ – $\mathrm{Sht}^{iter}_{(X-D)^I,K}$ classifies $\left\{ \begin{array}{l} (x_1,\dots,x_m) \in (X-D)^I \\ \mathcal{E} \xdashrightarrow[\text{mod. at } x_1]{} \mathcal{E}_1 \xdashrightarrow[\text{mod at } x_2]{} \cdots \dashrightarrow \mathcal{E}_m = {}^\sigma\mathcal{E} \end{array} \quad K\text{-level structure} \right\}$

$\downarrow \pi$

$\mathrm{Sat}_{\mu_\bullet} := \pi_* IC_{\mu_\bullet}$ – $\mathrm{Sht}_{(X-D)^I,K}$ classifies $\{ \underline{x} \in (X-D)^I,\ \mathcal{E} \xdashrightarrow{\text{isom away from } x_{1,\dots,m}} {}^\sigma\mathcal{E},\ K\text{-level structure} \}$

$\downarrow \mathrm{pr}$

$$(X-D)^I \ni \Delta(\bar\eta)$$

Define $H(I,W) := pr_!\big(\underset{\circlearrowleft \, \mathcal{H}k_K}{Sat_{\mu_\circ}}\big)\Big|_{\Delta(\bar\eta)} \circlearrowleft W^I_{F,D}$

<u>Basic question</u>: What is $H(I,W)$?

$G/k$ split, equipped w/ pinning $(B,T,e)$. Fix $\psi_0 : (F\backslash \mathbb{A}_F, +) \to \Lambda^\times$ Whittaker datum

<u>Local conjecture</u>

There is a natural <u>fully faithful</u> functor

$$A_x : \mathrm{Rep}_{f.g}(G(F_x)) \longrightarrow \mathrm{Coh}(\mathrm{Loc}_{W_{F_x},\hat G})$$

everything is derived

satisfying some natural compatibilities

generated by $\text{c-ind}_{K_x}^{G(F_x)} \Lambda$'s & colimits

(1) $\hat G = \mathbb{G}_m$, $\Gamma(\mathrm{Loc}_{W_{F_x},\mathbb{G}_m}) = C_c(F_x^\times, \Lambda)$

given by class field theory.

(2) For $K_x \subseteq G(F_x)$ open compact, put

$$A_{K_x} = A_x\big(\text{c-ind}_{K_x}^{G(F_x)} \Lambda\big) \in \mathrm{Coh}(\mathrm{Loc}_{W_{F_x},\hat G})$$

When $K_x$ is hyperspecial,

$$A_{K_x} = \mathcal{O}_{\mathrm{Loc}^{unr}_{W_{F_x},\hat G}}, \quad \mathrm{Loc}^{unr}_{W_{F_x},\hat G} = [\hat G/\mathrm{Ad}\,\hat G]$$

<u>Fully faithful</u> means

$$\mathrm{End}_{G(F_x)}\big(\text{c-ind}_{K_x}^{G(F_x)} \Lambda\big) = \mathrm{End}_{\mathrm{Loc}_{W_{F_x},\hat G}}(A_{K_x})$$

classical version $\quad \| \quad$

$\mathcal{H}k_{K_x}$ $\quad$ i.e. $\mathcal{H}k_{K_x} \circlearrowright A_{K_x}$

Rmk: When $K_x$ is hyperspecial,

<u>Global conjecture</u>

$$\begin{tikzcd} \tilde W \text{ - } \mathrm{Loc}_{W_{F,D},\hat G} \arrow[r, "f"] \arrow[dr, "W"'] & \prod_{x\in D} \mathrm{Loc}_{W_{F_x},\hat G} \arrow[d] \\ & {[\bullet/\hat G]} \end{tikzcd}$$

<u>Conjecture</u> $K \rightsquigarrow \boxtimes_x A_{K_x}$ over $\prod_{x\in D} \mathrm{Loc}_{W_{F_x},\hat G}$

$$H_K(I,W) = R\Gamma\big(\mathrm{Loc}_{W_{F,D},\hat G}, f^!(\boxtimes_x A_{K_x}) \otimes \tilde W\big)$$

$\mathcal{H}k_K \circlearrowright$, $W_{F,D} \circlearrowright$, $\mathcal{H}k_{K_x} \circlearrowright$, $W_{F,D} \circlearrowright$

This explains $\mathcal{H}k_{K_x}$-action when $x \in D$.

for $y \notin D$,

$$\begin{tikzcd} \mathrm{Loc}_{W_{F,D}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Loc}_{W_{F,D\cup\{y\}}} \arrow[d] \\ \mathrm{Loc}^{unr}_{W_{F_y}} \arrow[r, "i_*"] & \mathrm{Loc}_{W_{F_y}} \end{tikzcd}$$

So $R\Gamma(\mathrm{Loc}_{W_{F,D}}, f^!(\boxtimes_x A_{K_x}) \otimes \tilde W)$

$$\simeq R\Gamma\big(\mathrm{Loc}_{W_{F,D\cup\{y\}}}, f_y^!(\boxtimes_x A_{K_x} \boxtimes \mathcal{O}_{\mathrm{Loc}^{unr}_{W_{F_y}}}) \otimes \tilde W\big)$$

$\mathcal{H}k_y \circlearrowright \mathcal{O}_{\mathrm{Loc}^{unr}_{W_{F_y}}}$

There's a tautological action on $\tilde W$ by $W_{F,D}$

$$\mathrm{Loc}_{W_{F,D},\hat G} = [R_{W_{F,D},\hat G}/\hat G]$$

At each $\rho : W_{F,D} \to \hat G$, $\gamma \in W_{F,D}$ acts by

$$\mathrm{End}_{\mathrm{Loc}_{F_x}}\left(\mathcal{O}^{unr}_{\mathrm{Loc}_{F_x}}\right) = \Gamma\left(\mathrm{Loc}^{unr}_{F_x}, \mathcal{O}\right) = \Gamma\left(\hat{G}/\mathrm{Ad}\,\hat{G}, \mathcal{O}\right)$$

This gives classical Satake isom.

Rmk: The correct notion should be

$$R\mathrm{End}_{G(F_x)}\left(\text{c-ind}_{K_x}^{G(F_x)}\Lambda\right) \simeq R\mathrm{End}_{\mathrm{Loc}_{F_x}}\left(\mathcal{O}^{ur}_{\mathrm{Loc}_{F_x}}\right)$$

"derived Satake"

---

More cases of $\mathcal{A}_{K_x}$ ?

(3) Compatibility with parabolic induction

$M \overset{q}{\twoheadleftarrow} P \overset{\gamma}{\hookrightarrow} G \qquad \hat{M} \overset{\hat{q}}{\twoheadleftarrow} \hat{P} \overset{\hat{\gamma}}{\hookrightarrow} \hat{G}$

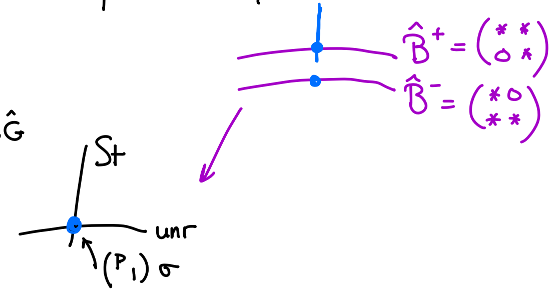
$$\begin{tikzcd}
\mathrm{Rep}_{f.g.}(M(F)) \arrow[r, "\mathcal{A}_M"] \arrow[d, "natural"] & \mathrm{Coh}(\mathrm{Loc}_{F_x,\hat{M}}) \arrow[d, "\hat{q}^!"] \\
\mathrm{Rep}_{f.g}(P(F)) \arrow[d, "\text{c-ind}_{P(F)}^{G(F)}(-)"] & \mathrm{Coh}(\mathrm{Loc}_{F_x,\hat{P}}) \arrow[d, "\hat{\gamma}_*"] \\
\mathrm{Rep}_{f.g}(G(F)) \arrow[r, "\mathcal{A}_G"] & \mathrm{Coh}(\mathrm{Loc}_{F_x,\hat{G}})
\end{tikzcd}$$

E.g. $M = T$. $\text{c-ind}_{T(\mathcal{O})}^{T(F)}\Lambda \longleftrightarrow \mathcal{O}_{\mathrm{Loc}^{ur}_{F_x,\hat{T}}}$

$$\begin{tikzcd}
\text{c-ind}_{B(F)}^{G(F)}\left(\text{c-ind}_{T(\mathcal{O})}^{T(F)}\Lambda\right) \arrow[d, phantom, "\cong"] \\
\text{c-ind}_{I_w}^{G(F)}\Lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Loc}^{unip}_{\hat{B}} \arrow[r] \arrow[d] \arrow[drr, dashed, "\hat{\gamma}"] & \mathrm{Loc}^{tame}_{\hat{B}} \arrow[d] \arrow[dr] & \\
\mathrm{Loc}^{unr}_{\hat{T}} \arrow[r, hook] & \mathrm{Loc}^{tame}_{\hat{T}} & \mathrm{Loc}^{tame}_{\hat{G}}
\end{tikzcd}$$

E.g. $\hat{G} = GL_2$, $\mathrm{Loc}^{unip}_{\hat{B}} = \left[R_{W_{F_x},\hat{B}}/\hat{B}\right] \leftarrow R_{W_{F_x},\hat{B}} \overset{\hat{B}}{\times} \hat{G} = \left\{(\sigma,\tau,\hat{B}') \,\middle|\, (\sigma,\tau)\in\hat{B}',\ \sigma\tau\sigma^{-1} = \tau^{p}\right\}$ $\quad \tau \sim \begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}$

$$\begin{tikzcd}
\left[R_{W_{F_x},\hat{B}}/\hat{B}\right] \arrow[d] & R_{W_{F_x},\hat{B}} \overset{\hat{B}}{\times} \hat{G} \arrow[l] \arrow[d] \\
\left[R_{W_{F_x},\hat{G}}/\hat{G}\right] & R_{W_{F_x},\hat{G}} \arrow[l]
\end{tikzcd}$$




$r_W \circ \rho(\gamma)$ on $W$.

---

Special fiber cycle interpreted: fix $x_0 \in D$ but $K_{x_0}$ hyp.

"Combining $\mathcal{A}_{x_0}$ with $\widetilde{W}$"

i.e. $\mathrm{Hom}_{\mathrm{Loc}^{unr}_{F_{x_0}}}\left(\mathcal{O}^{ur}\otimes\widetilde{W},\ \mathcal{O}^{ur}\otimes\widetilde{W}'\right)$

$\otimes$

$$R\Gamma\left(\mathrm{Loc}_{F,D},\ f^!\left(\boxtimes_{x\neq x_0}\mathcal{A}_x \otimes \left(\mathcal{A}_{x_0}\otimes\widetilde{W}\right)\right)\right)$$

$$\downarrow$$

$$R\Gamma\left(\mathrm{Loc}_{F,D},\ f^!\left(\boxtimes_{x\neq x_0}\mathcal{A}_x \otimes \left(\mathcal{A}_{x_0}\otimes\widetilde{W}'\right)\right)\right)$$

Expect to realize this by

$$\mathrm{Sht}_{x_0^I, W} \swarrow \ ? \ \searrow \mathrm{Sht}_{x_0^I, W'}$$

Conclusion $\hat{\gamma}_* \mathcal{O}_{\mathrm{Loc}^{\mathrm{ur}}_{\hat{B}}} \simeq \mathcal{O}_{\mathrm{Loc}^{\mathrm{unip}}_{\hat{G}}} \oplus \mathcal{O}_{\mathrm{Loc}^{\mathrm{ur}}_{\hat{G}}} =: \mathrm{Spr}_{\hat{G}} \in \mathrm{Coh}\left(\mathrm{Loc}^{\mathrm{tame}}_{\hat{G}}\right)$

Theorem (Hemo-Zhu) When $\Lambda = \overline{\mathbb{Q}}_\ell$, $\mathrm{Spr}_{\hat{G}}$ is concentrated in deg 0 and $\mathcal{H}k_{I_w} \simeq \mathrm{End}_{\mathrm{Loc}^{\mathrm{tame}}_{\hat{G}}}(\mathrm{Spr}_{\hat{G}})$

(4) Bernstein-Zelevinsky duality $\mathbb{D}^{\mathrm{Coh}} : \mathrm{Rep}(G(F_x), \Lambda) \longrightarrow \mathrm{Rep}(G(F_x), \Lambda)$

contravariant functor!

$$\pi \longmapsto R\mathrm{Hom}_{G(F_x)}(\pi, \mathcal{H}_G) \qquad \mathcal{H}_G = C_c^\infty(G(F_x), \Lambda)$$

$$\mathrm{c\text{-}ind}_K^{G(F_x)} \Lambda \longmapsto \mathrm{Hom}_{G(F_x)}\left(\mathrm{c\text{-}ind}_K^{G(F_x)} \Lambda, C_c^\infty(G(F_x), \Lambda)\right)$$

$$= \mathrm{Hom}_{K_x}(\Lambda, C_c^\infty(G(F_x), \Lambda))$$

$$\simeq \mathrm{c\text{-}ind}_{K_x}^{G(F_x)} \Lambda$$

$$\mathcal{A} \circ \mathbb{D}^{\mathrm{Coh}} = c^* \circ \mathbb{D}^{\mathrm{GS}} \circ \mathcal{A}$$

Grothendieck-Serre duality on $\mathrm{Loc}_{F_x}$

$c$ : Chevalley involution $\mathrm{G}(\hat{G}, \hat{B}, \hat{T})$ $\quad \lambda \mapsto -w_0(\lambda)$

(5) $\mathcal{A}\left(\mathrm{c\text{-}ind}_{U(F)}^{G(F)} \psi_0\right) = \mathcal{O}_{\mathrm{Loc}_{F_x, \hat{G}}}$

not f.g., need colimit

structure sheaf is NOT coherent b/c it's supported on $\infty$'ly many components

---

Geometric realization of $\mathrm{Hom}_{\mathrm{Loc}_{\hat{G}}}(\mathcal{A}_{I_w} \otimes \tilde{W}, \mathcal{A}_{I_w} \otimes \tilde{W}')$ (sketch)

Let $(G, X)$ be a Shimura datum, unram at $p$

$K_p = $ Iwahori, $G' = $ inner form of $G$, $G'(\mathbb{A}_f^p) \simeq G(\mathbb{A}_f^p) \supseteq K^p$, $G'(\mathbb{R})$ compact mod center

(do not assume that $V_\mu^{\mathrm{Tate}} \neq 0 \rightsquigarrow G' = J_b$ an inner form of $G_{\mathbb{Q}_p}$)

Theorem (Hemo-Zhu) There's a global Jacquet-Langlands transfer map

$$\mathrm{Hom}_{\mathrm{Loc}^{\mathrm{unip}}_{\hat{G}}}\left(\mathcal{A}_{J_b}, \tilde{V}_\mu \otimes \mathrm{Spr}\right) \longrightarrow \mathrm{Hom}_{\mathcal{H}_{K^p}}\left(C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K', \overline{\mathbb{Q}}_\ell), H^\cdot(\mathrm{Sh}_\mu, \Psi_\mu)\right)$$

One can imagine that this comes from taking categorical trace of

$$\mathrm{Coh}^{\mathrm{mon}}\left(\mathrm{St}^{\mathrm{unip}}_{\hat{G}}/\hat{G}\right) \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{cons}}\left(I_w \backslash LG / I_w\right)$$

$$\downarrow \mathrm{Tr}_\sigma \qquad\qquad \downarrow \text{pullback}$$

Key observation:

σ-twist cat. trace is precisely this $\rightsquigarrow$ $\mathrm{Coh}(\mathrm{Loc}^{\mathrm{unip}}_{\hat{G}}) \overset{\exists}{\dashrightarrow} \mathrm{Shv}(\mathrm{Sht}^{\mathrm{loc}}_{\mathrm{Iw}})$

## A "better formulation"

$$\begin{tikzcd}[column sep=small]
 & & \text{Correspondence} \arrow[dl] \arrow[dr] & \\
\mathcal{E} \dashrightarrow {}^{\sigma}\mathcal{E} \quad \mathrm{Sht}^{\mathrm{loc}} = [LG/\mathrm{Ad}_\sigma L^+G] \arrow[dr] & & \Diamond & {[LG/\mathrm{Ad}_\sigma L^+G]} \arrow[dl] \\
 & & {[LG/\mathrm{Ad}_\sigma LG] =: B(G)} &
\end{tikzcd}$$

$\uparrow$ σ-conjugacy classes of $G(\breve{\mathbb{Q}}_p)$

= "isocrystal up to isogeny"

General fact: $f: X \to Y$ proper representable morphism

$C := X \times_Y X$, then $f_!: D(X) \to D(Y)$ extends to a functor

$D(X) \subseteq D^C(X) \xrightarrow{f_!} D(Y)$. ← even fully faithful !!

$D^C(X)$ = category where morphisms are cohom corresp. suppored on $C$.

$$\begin{tikzcd}[column sep=small]
\mathcal{F} \arrow[dr, dash] & & C \arrow[dl, "p"'] \arrow[dr, "q"] & & \mathcal{G} \arrow[dl, dash] \\
 & X \arrow[dr, "f"'] & & X \arrow[dl, "f"] & \\
 & & Y & &
\end{tikzcd}$$

$\Rightarrow \mathrm{Hom}_C(p^*\mathcal{F}, q^!\mathcal{G}) = \mathrm{Hom}_Y(f_!\mathcal{F}, f_!\mathcal{G})$

b/c $f_* p_* \mathcal{H}om(p^*\mathcal{F}, q^!\mathcal{G})$

$$= f_* p_* \mathbb{D}(\mathbb{D}q^!\mathcal{G} \otimes p^*\mathcal{F}) = \mathbb{D} f_* p_* (q^*\mathbb{D}\mathcal{G} \otimes p^*\mathcal{F})$$
$$= \mathbb{D} f_* (\mathcal{F} \otimes p_* q^* \mathbb{D}\mathcal{G}) = \mathbb{D} f_* (\mathcal{F} \otimes f^* f_* \mathbb{D}\mathcal{G})$$
$$= \mathbb{D}(f_* \mathbb{D}\mathcal{G} \otimes f_*\mathcal{F}) = \mathcal{H}om(f_*\mathcal{F}, f_*\mathcal{G}).$$

Correct category: $\mathrm{Shv}(B(G))$

$B(G)(k)$ is a poset.

For each $b \in B(G)$, there's

$$\begin{tikzcd}
i_b : B(G)_b \arrow[r, hook, "j_b"] \arrow[d, equal] & B(G)_{\leq b} \arrow[r, hook, "i_{\leq b}"] & B(G) \\
{[*/J_b]} & B(G)_{<b} \arrow[u, hook, "i_b"'] &
\end{tikzcd}$$

Theorem (Hemo-Zhu)

(1) $\forall b \in B(G)$ $\mathrm{Shv}(B(G)_b, \Lambda) \simeq \mathrm{Rep}(J_b(F), \Lambda)$

(2) $\exists\, j_{b*}, j_{b!}\, j_b^!, i_{b*}, i_b^!, i_b^*$.

<u>Conjecture</u>: Fixing $(B, T, e)$ & $\psi : F \to \bar{\mathbb{Z}}_\ell^\times$

$\exists$ equiv. of $\infty$-categories $\mathbb{L}_G : \mathrm{Coh}(\mathrm{Loc}_{\hat{G}}) \simeq \mathrm{Shv}_{\mathrm{cons}}(B(G), \Lambda)$

<u>Theorem</u>. $\mathrm{Shv}^{\mathrm{unip}}_{\mathrm{cons}}(B(G), \bar{\mathbb{Q}}_\ell) = \{\mathcal{F}$ s.t. $\forall b \in B(G)$, $i_b^*\mathcal{F} \in \mathrm{Rep}(J_b(F), \bar{\mathbb{Q}}_\ell)$ are unipotent

i.e. $i_b^*\mathcal{F}$ appears in $\text{c-ind}_{P}^{J_b(F)} \pi$ $\leftarrow$ unipotent of Lusztig of $P(\mathbb{F}_p)$ $\}$

$\uparrow$ parahoric

$\wr\Vert$

$\mathrm{Coh}(\mathrm{Loc}^{\mathrm{unip}}_{\hat{G}})$

Moreover, each $i_{b*}(\text{c-ind}_{I_b}^{J_b} \Lambda) \longleftrightarrow \pi^{\mathrm{unip}}_{w_b *}(\mathcal{O}_{\widetilde{\mathrm{Loc}}^{\mathrm{unip}}_{\hat{G}, \omega}}(\lambda_b))$