Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 2. Arnold's Problem on Interval Exchange Permutations (after a joint work with V. Delecroix, E. Goujard, P. Zograf)

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Approach to Arnold's problem

Arnold's problem

2002-8. The (C, B, A)-permutation of the set $\{1, 2, \ldots, n\}$ transports to the last place the subset $A = \{1, 2, \ldots, a\}$ preceded by the transported set $B = \{a + 1, \ldots, a + b\}$ while the starting position is occupied by $C = \{a + b + 1, \ldots, n\}$.

Some of these $(n - 1)(n - 2)/2$ permutations permute *cyclically* (like the addition of a constant to the residues mod n), and some of these cyclic permutations are *transitive* (like the addition of the constant 1).

Find the proportion of both the cyclic and the transitive cyclic permutations among the (C, B, A) -permutations for large n.

More generally, starting from a permutation of k elements, one defines a permutation of the set $\{1, \ldots, n\}$ from its decomposition into k segments $\{a_i + 1, a_{i+1} - 1\}$. The problem is to study the statistics of the Young diagrams formed by the cycle lengths of the resulting permutations, for the case of large n and random decompositions of n into k parts.

Canonical suspension over an interval exchange

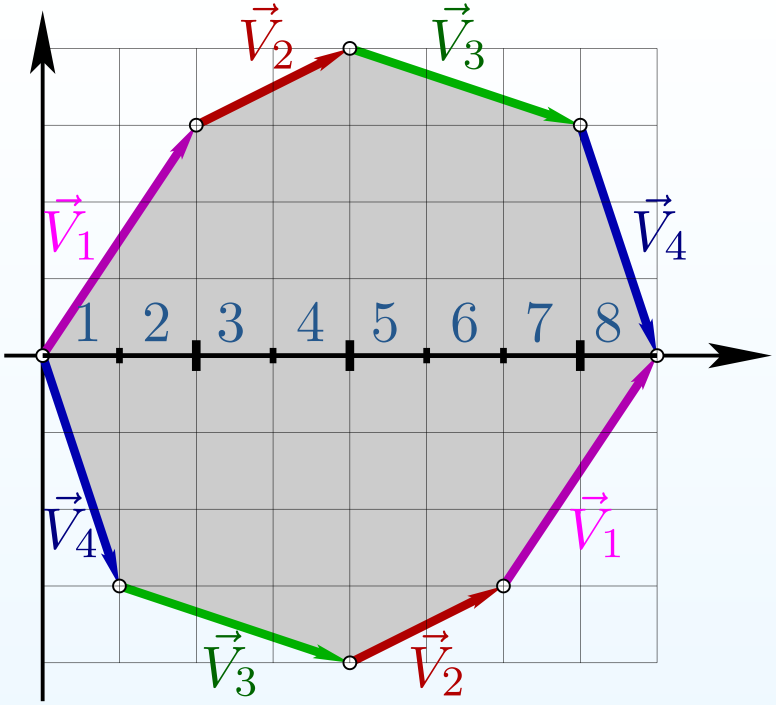
Consider an interval exchange transformation (iet) $T = (\pi, \lambda)$ of n subintervals, where T chops the interval $[0, \lambda_1 + \cdots + \lambda_n[$ into n consecutive subintervals of lengths $\lambda_1, \ldots, \lambda_n$ and places them on X preserving the orientation in the order $\pi^{-1}(1), \ldots, \pi^{-1}(n)$ without gaps or overlaps.

We always assume that T does not send consecutive intervals to consecutive intervals, that is $\pi(j+1) \neq \pi(j) + 1$ for $j = 1, \ldots, n-1$. (This condition is slightly weaker than the standard *nondegeneracy* condition of an iet). We also assume that π does not have nontrivial invariant subsets of the form $\{1, ..., k\}$ (otherwise T acts independently on two disjoint intervals).

Canonical suspension over an interval exchange

Consider a broken line in the plane formed from vectors $\vec{V}_j = (\lambda_j, \pi(j) - j)$ and another broken line starting from the same point and composed from the same vectors now placed in the order $\pi^{-1}(1), \ldots, \pi^{-1}(n)$ (as subintervals under exchange). Identifying the corresponding pairs of sides of the resulting polygon by parallel translations, we get a flat surface. The vertical flow on this surface realizes a suspension flow over the initial interval exchange. By convention, we mark the two points of the surface coming from the two vertices of the polygon corresponding to the endpoints of the broken lines.

Example of suspension



Suspension over an interval exchange transformation $T(\pi, \lambda)$ with parameters

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \lambda = (2, 2, 3, 1)\,.$$

The associated *interval exchange permutation* has the form

$$\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 6 & 7 & 3 & 4 & 1 & 2 \end{pmatrix}\,.$$

Vectors $\vec{V}_j = (\lambda_j, \pi(j) - j)$ of the canonical suspension have coordinates

$$\vec{V}_1 = (2, 3) \quad \vec{V}_2 = (2, 1) \quad \vec{V}_3 = (3, -1) \quad \vec{V}_4 = (1, -3)\,.$$

Bands of periodic trajectories

Definition. We say that cycles C_1 and C_2 of a permutation τ belong to the same *band*, if one can chose $k_1 \in C_1$ and $k_2 \in C_2$ such that

$$\tau^{(j)}(k_2) = \tau^{(j)}(k_1) + 1 \quad \text{or} \quad \tau^{(j)}(k_2) = \tau^{(j)}(k_1) - 1 \quad \text{for all } j \in \mathbb{Z}$$

and we consider the minimal equivalence relation induced by this property.

The permutation $(1, 10, 4, 7)(2, 11, 5, 8)(3, 6, 9)$ has two bands of cycles, where the cycles $(1, 10, 4, 7)$ and $(2, 11, 5, 8)$ belong to the same band. The permutation $(1, 9, 3, 5, 7)(2, 10, 4, 6, 8)$ has a single band of cycles.

We have seen that a (C, B, A)-permutation has a single band of cycles if and only if it is “cyclic” in the sense of Arnold; it has two band of cycles otherwise.

Important Observation. *Consider a permutation τ associated to an integer interval exchange transformation (π, λ), where $\lambda \in \mathbb{N}^n$. The number of bands of cycles of τ coincides with the number of maximal cylinders of the vertical suspension flow on the associated flat surface. (By convention, we mark the points on the surface (possibly a single point) corresponding to the endpoints of the interval if they are nonsingular points of the flat metric.)*

Why "bands of cycles" and not just cycles?

Fix a permutation π and consider statistics of the number of cycles of a random *interval exchange permutation* $\tau(\lambda, \pi)$ associated to an integer interval exchange transformation $T(\lambda, \pi)$ of the interval $[0, N[$ as $N \to \infty$. By "integer" interval exchange we call one with $\lambda \in \mathbb{N}^d$, where $d = \mathrm{Card}(\pi)$.

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2022). *For any permutation π the mean value of the number of cycles of a random interval exchange permutation $\tau(\lambda, \pi)$ is infinite.*
For any stratum of Abelian differentials, the mean value of the number of vertical (horizontal) bands of squares of a random square-tiled surface in this stratum is infinite.

Remark. Note that for numerous separatrix diagrams, the corresponding mean value for square-tiled surfaces representing these particular diagrams is finite!

The above Theorem explains why an adequate interpretation of Arnold's problem (the most general question about Young diagrams) suggests to consider *bands of cycles* and not cycles themselves.

Enhanced solution of Arnold's problem

Let π be a non degenerate irreducible permutation. Let $\mathcal{H}^{comp}(m_1,\dots,m_n)$ be a connected component of a stratum of Abelian differentials ambient for the canonical suspension over an interval exchange with a permutation π.

Let $d = \operatorname{Card}\pi$ be the number of elements in π.

Let $\operatorname{Vol}\mathcal{H}^{comp}(m_1,\dots,m_n)$ and $\operatorname{Vol}_k\mathcal{H}^{comp}(m_1,\dots,m_n)$ be respectively the Masur–Veech volume of the component and the contribution of k-cylinder square-tiled surfaces to this volume.

Let U be an open bounded set in $\mathbb{R}_+^d$. Denote by tU the set obtained from U by dilation with coefficient $t \in \mathbb{R}$. Denote by $IET(\pi, U, \varepsilon)$ and by $IET_k(\pi, U, \varepsilon)$ respectively the number of (π, λ)-integral interval exchange transformations such that $\lambda \in \mathbb{N}^d \cap \frac{1}{\varepsilon}U$ and the number of those of them, which have exactly k bands of periodic vertical trajectories.

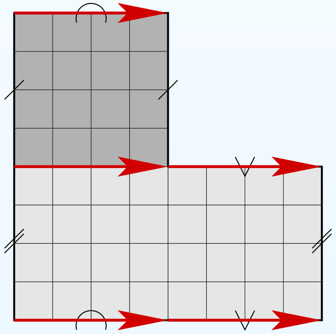
Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020) For any π, U, k as above one has

$$\lim_{\varepsilon\to+0}\frac{IET_k(\pi, I, \varepsilon)}{IET(\pi, I, \varepsilon)} = \frac{\operatorname{Vol}_k\mathcal{H}^{comp}(m_1,\dots,m_n)}{\operatorname{Vol}\mathcal{H}^{comp}(m_1,\dots,m_n)}\,.$$

Explicit answers for low dimensional strata

Critical graphs (separatrix diagrams)

Note that all leaves of the horizontal (vertical) foliation on a square-tiled surface are closed. The *critical graph* Γ (*separatrix diagram*) is the union of all horizontal critical leaves. Vertices of Γ are represented by the conical points; the edges of Γ are formed by horizontal saddle connections.



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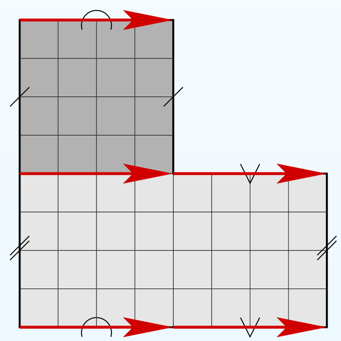
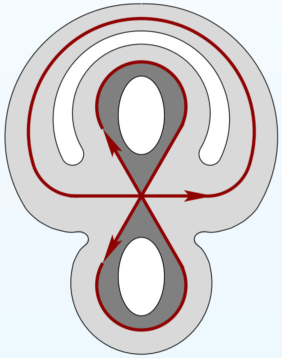




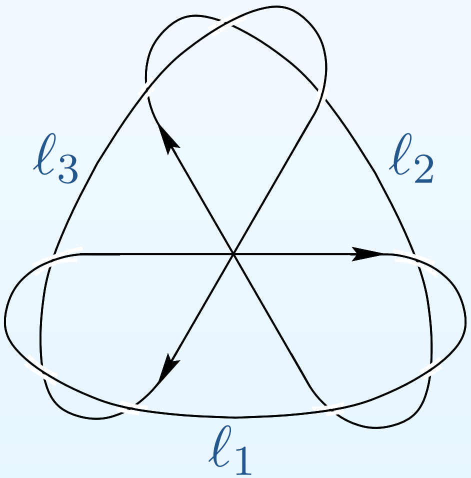
A critical graph Γ is an *oriented ribbon graph* endowed with the following structure:

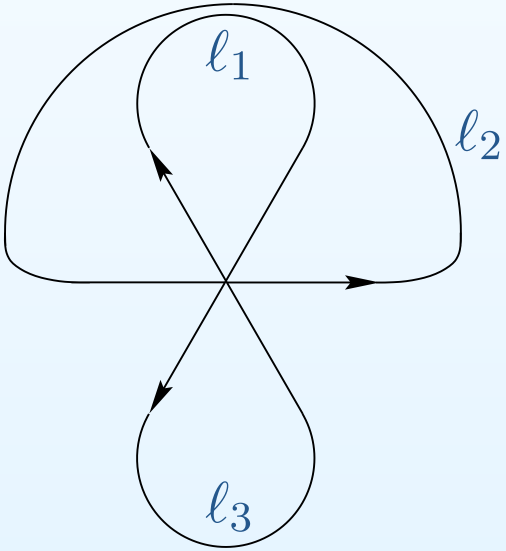
1. The orientation of edges at any vertex is alternated with respect to the cyclic order of edges at this vertex.

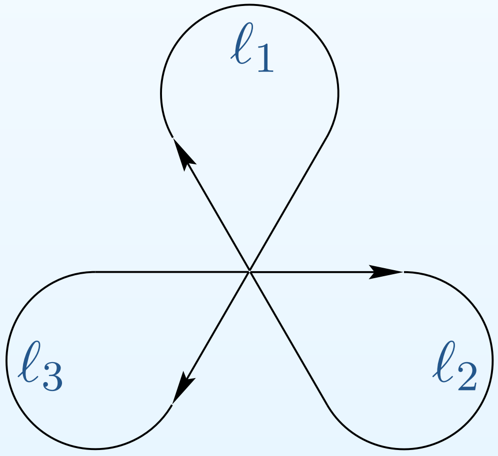
2. The complement $S - \Gamma$ is a finite disjoint union of flat cylinders foliated by oriented circles. Thus, the set of boundary components of the ribbon graph is decomposed into pairs: to each pair of boundary components we glue a cylinder, and there is one positively oriented and one negatively oriented boundary component in each pair.

Realizable separatrix diagrams

Note, however, that not all ribbon graphs as above correspond to actual flat surfaces. A flat metric endows saddle connections with positive lengths ℓ_i. The left graph is realizable for any lengths ℓ_1, ℓ_2, ℓ_3. The middle one — only when $\ell_1 = \ell_3$. The rightmost one is never realizable: pairs of boundary components bounding the same cylinder have to have equal length, and we cannot find a pair for the component of length $\ell_1 + \ell_2 + \ell_3$.

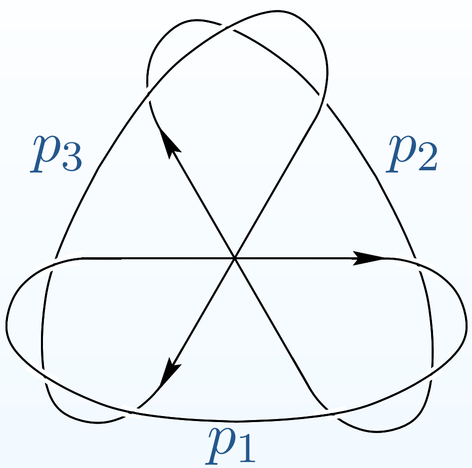






Lemma. *For any realizable separatrix diagram, the set of all square-tiled surfaces sharing this diagram provides a nontrivial contribution to the volume of the corresponding stratum.*

Volume computation for $\mathcal{H}(2)$: the one-cylinder diagram

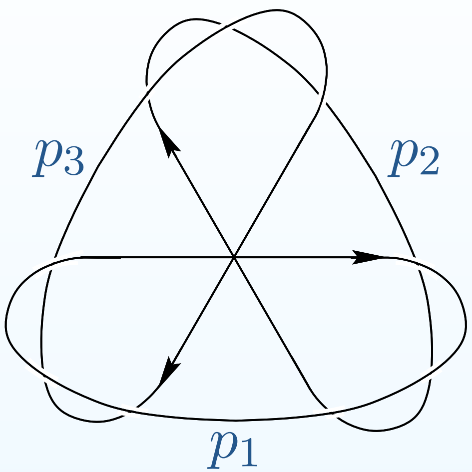


$$\frac{1}{3}\sum_{\substack{p_1,p_2,p_3,h\in\mathbb{N}\\(p_1+p_2+p_3)h\le N}}(p_1+p_2+p_3)\;\approx\;\frac{1}{3}\sum_{\substack{w,h\in\mathbb{N}\\w\cdot h\le N}}w\cdot\frac{w^2}{2}=\frac{1}{6}\sum_{\substack{w,h\in\mathbb{N}\\w\le\frac{N}{h}}}w^3$$

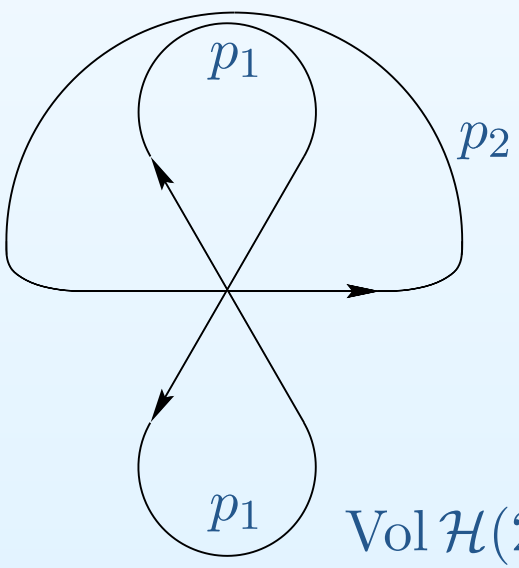
$$\approx\frac{1}{6}\sum_{h\in\mathbb{N}}\frac{1}{4}\cdot\left(\frac{N}{h}\right)^4=\frac{N^4}{24}\cdot\sum_{h\in\mathbb{N}}\frac{1}{h^4}\;=\;\frac{N^4}{24}\cdot\zeta(4)=\frac{N^4}{24}\cdot\frac{\pi^4}{90}\,.$$

Note that the contributions of surfaces for which our single maximal cylinder is composed of $1, 2, 3, \ldots$ parallel bands of squares correspond to the summands $\frac{1}{1^4}, \frac{1}{2^4}, \frac{1}{3^4}, \ldots$ of the factor $\zeta(4)=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots$.

Volume computation for $\mathcal{H}(2)$



$$\frac{1}{3} \sum_{\substack{p_1,p_2,p_3,h\in\mathbb{N} \\ (p_1+p_2+p_3)h\le N}} (p_1+p_2+p_3) \approx \frac{N^4}{24}\cdot\zeta(4)$$



$$\sum_{\substack{p_1,p_2,h_1,h_2 \\ p_1h_1+(p_1+p_2)h_2\le N}} p_1(p_1+p_2)$$

$$= \frac{N^4}{24}\big[2\cdot\zeta(1,3)+\zeta(2,2)\big] = \frac{N^4}{24}\cdot\frac{5}{4}\cdot\zeta(4)$$

$$\operatorname{Vol}\mathcal{H}(2) = \lim_{N\to\infty}\frac{2\cdot 4}{N^4}\cdot\text{(Number of all surfaces)} = \frac{3}{4}\zeta(4) = \frac{\pi^4}{120}$$

Answer to Arnold's problem for permutations of 4 elements

- The asymptotic proportion of *transitive* permutations among all $(4,3,2,1)$-permutations equals $\frac{4}{9\zeta(4)} = \frac{40}{\pi^4}$.

- The asymptotic proportion of "cyclic" permutations (in Arnold's sense, i.e. of 1-band permutations in our terminology) among all $(4,3,2,1)$-permutations equals $\frac{4}{9}$.

- The proportion of 2-band permutations is $\frac{5}{9}$. The larger number of bands is not realizable for $(4,3,2,1)$-permutations.

- Permutations $(2,4,1,3)$, $(2,4,3,1)$, $(3,1,4,2)$, $(3,2,4,1)$, $(4,1,3,2)$, $(4,2,1,3)$, $(4,3,2,1)$ in the Rauzy class of $\mathcal{H}(2)$ share the same proportions.

- All the remaining permutations of 4 elements are either degenerate or reducible.

- A random square-tiled surface in the stratum $\mathcal{H}(2)$ has a single maximal vertical cylinder with probability $\frac{4}{9}$ and two maximal cylinders with probability $\frac{5}{9}$.

Multiple zeta-values

Define

$$\zeta(s_1, s_2, \dots, s_k) = \sum_{n_1,\dots,n_k \geq 1} \frac{1}{n_1^{s_1}(n_1+n_2)^{s_2}\dots(n_1+\dots+n_k)^{s_k}}\,.$$

Multiple zeta-values (MZV) are values of $\zeta(s_1, s_2, \dots, s_k)$ at positive integers $s_j \in \mathbb{N}$, where $s_k \geq 2$. For example

$$\zeta(2) = \frac{\pi^2}{6}; \qquad \zeta(4) = \frac{\pi^4}{90}; \quad \dots \quad \zeta(2n) = \frac{p}{q}\pi^{2n}, \quad \text{where } p, q \in \mathbb{N}\,.$$

Conjecturally $\pi, \zeta(3), \zeta(5), \dots$ are algebraically independent over $\mathbb{Q}$.

Multiple zeta values satisfy numerous relations. The ones, which we used, namely

$$\zeta(1,3) = \frac{1}{4}\zeta(4); \qquad \zeta(2,2) = \frac{3}{4}\zeta(4) \quad ,$$

were already known to L. Euler.

Contributions $\operatorname{Vol}_k \mathcal{H}(3,1)$ of k-cylinder surfaces to $\operatorname{Vol}\mathcal{H}(3,1)$

$$\operatorname{Vol}_1 \mathcal{H}(3,1) = \frac{\zeta(7)}{15}$$

$$\operatorname{Vol}_2 \mathcal{H}(3,1) = \frac{55\,\zeta(1,6) + 29\,\zeta(2,5) + 15\,\zeta(3,4) + 8\,\zeta(4,3) + 4\,\zeta(5,2)}{45}$$

$$\begin{aligned}\operatorname{Vol}_3 \mathcal{H}(3,1) = \frac{1}{90}\Big(& 12\,\zeta(6) - 12\,\zeta(7) + 48\,\zeta(4)\,\zeta(1,2) + 48\,\zeta(3)\,\zeta(1,3)\\
&+ 24\,\zeta(2)\,\zeta(1,4) + 6\,\zeta(1,5) - 250\,\zeta(1,6) - 6\,\zeta(3)\,\zeta(2,2)\\
&- 5\,\zeta(2)\,\zeta(2,3) + 6\,\zeta(2,4) - 52\,\zeta(2,5) + 6\,\zeta(3,3) - 82\,\zeta(3,4)\\
&+ 6\,\zeta(4,2) - 54\,\zeta(4,3) + 6\,\zeta(5,2) + 120\,\zeta(1,1,5) - 30\,\zeta(1,2,4)\\
&- 120\,\zeta(1,3,3) - 120\,\zeta(1,4,2) - 54\,\zeta(2,1,4) - 34\,\zeta(2,2,3)\\
&- 29\,\zeta(2,3,2) - 88\,\zeta(3,1,3) - 34\,\zeta(3,2,2) - 48\,\zeta(4,1,2)\Big)\end{aligned}$$

$$\operatorname{Vol}_4 \mathcal{H}(3,1) = \frac{2\zeta(2)}{45}\Big(\zeta(4) - \zeta(5) + \zeta(1,3) + \zeta(2,2) - \zeta(2,3) - \zeta(3,2)\Big)\,.$$

After simplification

Multiple zeta values satisfy numerous relations. After simplification (which is now accessible through a SAGE package) we get

$$\operatorname{Vol}_1 \mathcal{H}(3,1) = 1/15 \cdot \zeta(7)$$
$$\operatorname{Vol}_2 \mathcal{H}(3,1) = -7/135 \cdot \zeta(1,6) + 1/135 \cdot \zeta(2,5) + 23/135 \cdot \zeta(7)$$
$$\operatorname{Vol}_3 \mathcal{H}(3,1) = -2/15 \cdot \zeta(1,6) - 2/45 \cdot \zeta(2,5) + 1/5 \cdot \zeta(6) - 4/45 \cdot \zeta(7)$$
$$\operatorname{Vol}_4 (\mathcal{H}(3,1) = 5/27 \cdot \zeta(1,6) + 1/27 \cdot \zeta(2,5) + 7/45 \cdot \zeta(6) - 4/27 \cdot \zeta(7)$$

Conjecturally, multiple zeta values involved in these simplified expressions are linearly independent over rational numbers. However, the total contribution is a rational multiple of π^{2g} in accordance with the general result by A. Eskin and A. Okounkov, 2001:

$$\operatorname{Vol} \mathcal{H}(3,1) = \operatorname{Vol}_1 \mathcal{H}(3,1) + \cdots + \operatorname{Vol}_4 \mathcal{H}(3,1) = \frac{16}{45}\zeta(6) = \frac{16}{42525}\pi^6$$

Conjecture on Delecroix sums

Conjecture (V. Delecroix, A. Zorich). *For any connected component of any stratum $\mathcal{H}(m_1, \ldots, m_n)$ of Abelian differentials and for any positive integer k, the contribution $\operatorname{Vol}_k \mathcal{H}^{comp}(m_1, \ldots, m_n)$ of k-cylinder square-tiled surfaces to the Masur-Veech volume of the component of the stratum is a is a linear combination of multiple zeta values with rational coefficients.*

For $k = 1$ this fact is elementary; for $k = 2$ it is relatively easy to prove; for $k = 3$ it is already a nontrivial theorem due to B. Allombert and V. Delecroix. These three rigorous results valid for all strata combined with further direct computations in small genera strongly corroborate to the conjecture.

This (conjectural) form of contribution of k-cylinder square-tiled surfaces indicates that they might have geometric meaning which we do not understand yet. It is a great challenge to interpret these contributions as certain periods and relate them to cycles in the moduli space. The arithmetic properties of the "Delecroix sums" which appear as contributions of individual separatrix diagrams to the Masur-Veech volumes are interesting by themselves.

$$
\begin{aligned}
\mathrm{Vol}_5(\mathcal{H}(2,1,1)) = \frac{1}{1260}\cdot\Big(&10\zeta(2)\zeta(4)-30\zeta(2)\zeta(5)+77\zeta(6)+20\zeta(2)\zeta(6)-231\zeta(7)\\
&+154\zeta(8)+40\zeta(3)\zeta(1,2)+64\zeta(2)\zeta(1,3)+32\zeta(3)\zeta(1,3)\\
&-24\zeta(4)\zeta(1,3)+138\zeta(2)\zeta(1,4)-96\zeta(3)\zeta(1,4)-326\zeta(1,5)\\
&-240\zeta(2)\zeta(1,5)-1650\zeta(1,6)+2736\zeta(1,7)+17\zeta(2)\zeta(2,2)\\
&+32\zeta(3)\zeta(2,2)-12\zeta(4)\zeta(2,2)+27\zeta(2)\zeta(2,3)\\
&-56\zeta(3)\zeta(2,3)+26\zeta(2,4)-54\zeta(2)\zeta(2,4)-805\zeta(2,5)\\
&+1146\zeta(2,6)-14\zeta(2)\zeta(3,2)-12\zeta(3)\zeta(3,2)+54\zeta(3,3)\\
&+16\zeta(2)\zeta(3,3)-407\zeta(3,4)+524\zeta(3,5)+96\zeta(4,2)\\
&+12\zeta(2)\zeta(4,2)-268\zeta(4,3)+234\zeta(4,4)-272\zeta(5,2)\\
&+176\zeta(5,3)+160\zeta(6,2)+108\zeta(1,1,4)-468\zeta(1,1,5)\\
&+240\zeta(1,1,6)-22\zeta(1,2,3)-558\zeta(1,2,4)+576\zeta(1,2,5)\\
&-42\zeta(1,3,2)-304\zeta(1,3,3)+336\zeta(1,3,4)-258\zeta(1,4,2)\\
&+264\zeta(1,4,3)+336\zeta(1,5,2)+6\zeta(2,1,3)-282\zeta(2,1,4)\\
&+336\zeta(2,1,5)+27\zeta(2,2,2)-454\zeta(2,2,3)+432\zeta(2,2,4)\\
&-365\zeta(2,3,2)+400\zeta(2,3,3)+330\zeta(2,4,2)-40\zeta(3,1,2)\\
&-116\zeta(3,1,3)+120\zeta(3,1,4)-240\zeta(3,2,2)+244\zeta(3,2,3)\\
&+192\zeta(3,3,2)+48\zeta(4,1,3)+108\zeta(4,2,2)\Big)
\end{aligned}
$$

After simplification

After simplification (which is now accessible through a SAGE package) we get

$$\mathrm{Vol}_1\,\mathcal{H}(2,1,1) = 7/180\cdot\zeta(8)$$

$$\mathrm{Vol}_2\,\mathcal{H}(2,1,1) = -2/63\cdot\zeta(1,7)+1/63\cdot\zeta(2,6)+1/36\cdot\zeta(7)+59/756\cdot\zeta(8)$$

$$\mathrm{Vol}_3\,\mathcal{H}(2,1,1) = 8/63\cdot\zeta(1,1,6)-1/378\cdot\zeta(1,6)-26/63\cdot\zeta(1,7)$$
$$+61/3780\cdot\zeta(2,5)-4/63\cdot\zeta(2,6)+953/3780\cdot\zeta(7)-1213/7560\cdot\zeta(8)$$

$$\mathrm{Vol}_4\,\mathcal{H}(2,1,1) = -16/63\cdot\zeta(1,1,6)-365/756\cdot\zeta(1,6)+58/63\cdot\zeta(1,7)$$
$$-187/1890\cdot\zeta(2,5)+5/63\cdot\zeta(2,6)+1/18\cdot\zeta(6)+983/3780\cdot\zeta(7)-83/280\cdot\zeta(8)$$

$$\mathrm{Vol}_5\,\mathcal{H}(2,1,1) = 8/63\cdot\zeta(1,1,6)+367/756\cdot\zeta(1,6)-10/21\cdot\zeta(1,7)$$
$$+313/3780\cdot\zeta(2,5)-2/63\cdot\zeta(2,6)+7/36\cdot\zeta(6)-2041/3780\cdot\zeta(7)+257/756\cdot\zeta(8)$$

$$\mathrm{Vol}\,\mathcal{H}(2,1,1) = 1/4\cdot\zeta(6)$$

Conjecturally, multiple zeta values involved in these simplified expressions are linearly independent over rational numbers. Once again, the total contribution is a rational multiple of π^{2g} in accordance with the general result by A. Eskin and A. Okounkov, 2001:

$$\mathrm{Vol}\,\mathcal{H}(2,1,1) = \mathrm{Vol}_1\,\mathcal{H}(2,1,1)+\cdots+\mathrm{Vol}_5\,\mathcal{H}(2,1,1) = \frac{1}{4}\zeta(6) = \frac{\pi^6}{3780}$$

There is an awful moment in popular books on cosmic theories (that breezily begin with plain straightforward chatty paragraphs) when there suddenly start to sprout mathematical formulas, which immediately blind one's brain. We do not go as far as that here.

V. Nabokov

General answer for the asymptotic proportion of transitive and "cyclic" interval exchange permutations

Cyclic versus transitive interval exchange permutations

Let π be an irreducible non-degenerate permutation. Denote by $p_{tr}(\pi)$ and by $p_{cyc}(\pi)$ the asymptotic proportions of respectively transitive and "cyclic" (in the sense of Arnold) permutations among all π-interval exchange permutations.

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020)
The quantities $p_{cyc}(\pi)$ and $p_{tr}(\pi)$ satisfy the following relation:

$$p_{cyc}(\pi) = p_{tr}(\pi) \cdot \zeta(d)\,,$$

where d is the number of elements in π.

Remark. Note that $\zeta(d) \to 1$ as $d \to +\infty$, and the convergence is very rapid. The above formula shows, in particular, that for permutations of sufficiently large number of elements the quantities $p_{cyc}(\pi)$ and $p_{tr}(\pi)$ become basically indistinguishable: majority of "cyclic" (in the sense of Arnold) permutations are necessarily transitive.

Characters of the symmetric group

Recall that a representation ρ of the symmetric group S_n is a homomorphism $\rho : S_n \to \mathrm{GL}(V)$ where V is a finite dimensional complex vector space. The simplest example is given by the permutation action of S_n on coordinates in $\mathbb{C}^n$. This action leaves invariant the 1-dimensional subspace generated by the sum $\vec{e}_1 + \vec{e}_2 + \ldots + \vec{e}_n$ of the vectors of the basis and the $(n-1)$-dimensional subspace $W_n := \{\sum x_i \vec{e}_i : \sum x_i = 0\}$, where $\vec{e}_i$ denotes the elements of the standard basis of $\mathbb{C}^n$. The representation $\mathbf{St}_n$ induced on W_n is irreducible (i.e. it does not contain non-trivial invariant subspaces).

Now define the characters of the exterior powers of the representation $\mathbf{St}_n$

$$\chi_j(g) := \mathrm{tr}(g, \pi_j) \qquad \pi_j := \wedge^j(\mathbf{St}_n) \qquad (0 \le j \le n-1)\,.$$

Answer to Arnold's problem for general permutations

Let $\mathcal{H}^{comp}(m_1,\dots,m_n)$ be the component of the stratum associated to the canonical suspension over π. Let $\operatorname{Vol}\mathcal{H}^{comp}(m_1,\dots,m_n)$ and $\operatorname{Vol}_1\mathcal{H}^{comp}(m_1,\dots,m_n)$ be respectively the Masur–Veech volume of this component and the contribution of 1-cylinder square-tiled surfaces to this volume.

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020) *A random π-interval exchange permutation is cyclic with probability*

$$p_{cyc}(\pi) = \frac{\operatorname{Vol}_1\mathcal{H}^{comp}(m_1,\dots,m_n)}{\operatorname{Vol}\mathcal{H}^{comp}(m_1,\dots,m_n)}\,,$$

Let $d = \dim\mathcal{H}(m_1,\dots,m_n) = 1+\sum_{j=1}^{n}(m_j+1)$ be the number of elements in π. Let $\nu \in S_{d-1}$ be any permutation which decomposes into cycles of lengths $(m_1+1),\dots,(m_n+1)$, and let μ_k be the multiplicity of the entry k in the multiset $\{m_1,\dots,m_n\}$. Then

$$\operatorname{Vol}_1\mathcal{H}(m_1,\dots,m_n) = \frac{2\zeta(d)}{(d-2)!}\cdot\prod_k\frac{1}{(k+1)^{\mu_k}}\cdot\sum_{j=0}^{d-1} j!\,(d-1-j)!\,\chi_j(\nu)\,.$$

Case of hyperelliptic components

When the permutation π corresponds to a hyperelliptic connected component, the formulae become much more explicit.

Theorem (J. Athreya, A. Eskin, A. Zorich, 2016)

$$\operatorname{Vol}\mathcal{H}^{hyp}(2g-2) = \frac{2\pi^{2g}}{(2g+1)!}\cdot\frac{(2g-3)!!}{(2g-2)!!}.$$

$$\operatorname{Vol}\mathcal{H}^{hyp}(g-1,g-1) = \frac{4\pi^{2g}}{(2g+2)!}\cdot\frac{(2g-2)!!}{(2g-1)!!}.$$

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020)

$$\frac{\operatorname{Vol}_1\mathcal{H}^{hyp}(2g-2)}{\operatorname{Vol}\mathcal{H}^{hyp}(2g-2)} = \frac{\zeta(2g)}{\pi^{2g}}\cdot 2g(2g+1)\cdot\frac{(2g-2)!!}{(2g-3)!!}.$$

$$\frac{\operatorname{Vol}_1\mathcal{H}^{hyp}(g-1,g-1)}{\operatorname{Vol}\mathcal{H}^{hyp}(g-1,g-1)} = \frac{\zeta(2g+1)}{2\pi^{2g}}\cdot(2g+1)(2g+2)\cdot\frac{(2g-1)!!}{(2g-2)!!}.$$

Case of even and odd components

Theorem (V. Delecroix, 2013) *For any stratum, for which all degrees of zeroes are even, one has*

$$\begin{aligned}\operatorname{Vol}_1 \mathcal{H}^{odd}(m_1,\dots,m_n) - \operatorname{Vol}_1 \mathcal{H}^{even}(m_1,\dots,m_n)\\ - (-1)^{\varphi} \operatorname{Vol}_1 \mathcal{H}^{hyp}(m_1,\dots,m_n) = \frac{\zeta(d)}{2^{g-2}} \cdot \prod_k \frac{1}{(k+1)^{\mu_k}}\,.\end{aligned}$$

Combining this formula with the formula for the total contribution

$$\begin{aligned}\operatorname{Vol}_1 \mathcal{H}(m_1,\dots,m_n) = \operatorname{Vol}_1 \mathcal{H}^{hyp}(m_1,\dots,m_n)\\ + \operatorname{Vol}_1 \mathcal{H}^{odd}(m_1,\dots,m_n) + \operatorname{Vol}_1 \mathcal{H}^{even}(m_1,\dots,m_n)\end{aligned}$$

and with a formula for $\operatorname{Vol}_1 \mathcal{H}^{hyp}(m_1,\dots,m_n)$ presented at the previous slide, we compute the individual contributions $\operatorname{Vol}_1 \mathcal{H}^{odd}(m_1,\dots,m_n)$ and $\operatorname{Vol}_1 \mathcal{H}^{even}(m_1,\dots,m_n)$ separately.

Correction due to hyperelliptic components

For strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(2k,2k)$ the above formula includes the contribution of hyperelliptic 1-cylinder square-tiled surfaces to the Masur-Veech volume. Abelian differentials in these hyperelliptic components also have certain parity φ of the spin structure. This parity is present in the sign $(-1)^\varphi$ of the contribution of the hyperelliptic component. This parity is computed using the formula below.

Theorem (M. Kontsevich, A. Zorich, 2003) *Parity of the spin structure determined by an Abelian differential from the hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ equals*

$$\varphi(\mathcal{H}^{hyp}(2g-2)) \equiv \left[\frac{g+1}{2}\right] \pmod 2 .$$

Parity of the spin structure of the hyperelliptic component $\mathcal{H}^{hyp}(g-1,g-1)$ for odd genera g equals

$$\varphi\left(\mathcal{H}^{hyp}(g-1,g-1)\right) \equiv \left(\frac{g+1}{2}\right) \pmod 2 \quad for\ \ odd\ \ g\,.$$

Principal and minimal strata

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020) *The contribution of 1-cylinder square-tiled surfaces to the Masur–Veech volume of the principal and of the minimal stratum of Abelian differentials satisfy*

$$\operatorname{Vol}_1 \mathcal{H}(1^{2g-2}) = \frac{\zeta(4g-3)}{4g-2} \cdot \frac{4}{2^{2g-2}}$$

$$\operatorname{Vol}_1 \mathcal{H}(2g-2) = \frac{\zeta(2g)}{2g} \cdot \frac{4}{2g-1}$$

In particular, we recover the value

$$\operatorname{Vol}_1 \mathcal{H}(2) = \frac{\zeta(4)}{4} \cdot \frac{4}{3} = \frac{\zeta(4)}{3},$$

which we have already seen from the direct computation.

Large genus asymptotics

Asymptotics of Masur–Veech volume in large genera

Let $\mathbf{m} = (m_1, \ldots, m_n)$ be an unordered partition of an even number $2g-2$, $|\mathbf{m}| = m_1 + \cdots + m_n = 2g-2$. Denote by Π_{2g-2} the set of all partitions. The following result was conjectured by A. Eskin and A. Zorich in 2003 and proved independently by A. Aggarwal and by D. Chen, M. Möller, A. Sauvaget, D. Zagier in 2020.

Asymptotics of Volumes (A. Aggarwal, 2020; D. Chen, M. Möller, A. Sauvaget, D. Zagier, 2020). *For any* $\mathbf{m} \in \Pi_{2g-2}$ *one has*

$$\operatorname{Vol}\mathcal{H}(m_1, \ldots, m_n) = \frac{4}{(m_1+1)\cdot \cdots \cdot (m_n+1)} \cdot (1+\varepsilon(\mathbf{m}))\,,$$

where $|\varepsilon(\mathbf{m})| \leq \dfrac{\mathrm{const}}{\sqrt{g}}$.

Let

$$\frac{\operatorname{Vol}\mathcal{H}^{even}(2k_1, \ldots, 2k_n)}{\operatorname{Vol}\mathcal{H}^{odd}(2k_1, \ldots, 2k_n)} = 1 + \delta(\mathbf{k})\,.$$

Then $\max_{\mathbf{k}\in\Pi_{g-1}} |\delta(\mathbf{k})| \to 0$ *as* $g \to +\infty$*.*

Volume contribution of 1-cylinder surfaces

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich, 2020) *The contribution $\operatorname{Vol}_1 \mathcal{H}(m_1, \dots, m_n)$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume of any stratum of Abelian differentials satisfies*

$$\frac{\zeta(d)}{d+1} \cdot \frac{4}{(m_1+1)\dots(m_n+1)} \le \operatorname{Vol}_1 \mathcal{H}(m_1, \dots, m_n) \le \frac{\zeta(d)}{d - \frac{10}{29}} \cdot \frac{4}{(m_1+1)\dots(m_n+1)}\,,$$

where $d = \dim_{\mathbb{C}} \mathcal{H}(m_1, \dots, m_n) = m_1 + \dots + m_n + n + 1$.

Let

$$\frac{\operatorname{Vol}_1 \mathcal{H}^{even}(2k_1, \dots, 2k_n)}{\operatorname{Vol}_1 \mathcal{H}^{odd}(2k_1, \dots, 2k_n)} = 1 + \Delta(k)\,.$$

Then $\max_{\mathbf{k} \in \Pi_{g-1}} |\Delta(\mathbf{k})| \to 0$ *as* $g \to +\infty$.

Arnold's problem for large number of intervals

Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Zorich) *The relative contribution $p_1(\mathcal{H}^{comp}(m_1,\dots,m_n))$ of 1-cylinder square-tiled surfaces to the Masur–Veech volume of any nonhyperelliptic component of any stratum satisfies*

$$d \cdot p_1(\mathcal{H}^{comp}(m_1,\dots,m_n)) \to 1 \ \text{ as } \ g \to +\infty\,,$$

where $d = \dim_{\mathbb{C}} \mathcal{H}^{comp}(m_1,\dots,m_n) = m_1 + \dots + m_n + n + 1$ and convergence is uniform for all strata in genus g.

Let $\pi \in S_k$ be a non degenerate irreducible permutation in the complement of the Rauzy classes of hyperelliptic components. The probability $p_{tr}(\pi)$ that a random π-interval exchange transformation is transitive satisfies

$$k \cdot p_{tr}(\pi) \to 1 \ \text{ as } \ k \to +\infty\,,$$

where convergence is uniform for all permutations in S_k satisfying the abovementioned conditions.

Recall that for $\pi \in S_k$ as above one has $\frac{p_{cyc}(\pi)}{p_{tr}(\pi)} = \zeta(k)$ and that $\zeta(k) \to 1$ as $k \to +\infty$, where the convergence is very rapid.

Conjectural universality in large genera

Conjecture (V. Delecroix, E. Goujard, P. Zograf, A. Zorich). *Let $x > 0$. Let $\mathcal{C}$ be a non-hyperelliptic connected component of a stratum of Abelian differentials. Let $p_k(\mathcal{C})$ denote the probability that a random Abelian square-tiled surface in $\mathcal{C}$ has k cylinders. Then uniformly for k in $\{0, 1, \ldots, \lfloor x \log(\dim_{\mathbb{C}} \mathcal{C}) \rfloor\}$ and uniformly in $\mathcal{C}$ such that $\dim \mathcal{C} \to \infty$*

$$p^{Ab}_{k+1}(\mathcal{C}) = \frac{1}{\dim_{\mathbb{C}} \mathcal{C}} \cdot \frac{(\log \dim_{\mathbb{C}} \mathcal{C})^k}{k!} \cdot \left(\frac{1}{\Gamma\left(1 + \frac{k}{\log \dim_{\mathbb{C}} \mathcal{C}}\right)} + o(1) \right).$$

In plain words, the Conjecture claims that the statistics $p^{Ab}_k(\mathcal{C})$ becomes practically indistinguishable from the statistics of the number of disjoint cycles in the cycle decomposition of a random permutation in $S_{\dim_{\mathbb{C}} \mathcal{C}}$, with respect to the uniform probability measure on the symmetric group of $\dim_{\mathbb{C}} \mathcal{C}$ elements.

Numerical evidence

The Conjecture is based on analyzing huge experimental data. We experimentally collected statistics of the number $K_{\mathcal{C}}(S)$ of maximal horizontal cylinders in cylinder decompositions of random square-tiled surfaces in about 30 connected components $\mathcal{C}$ of strata in genera from 40 to $10\,000$. In particular, the least squares linear approximation for components $\mathcal{C}$ of dimension $\dim_{\mathbb{C}} \mathcal{C}$ between 400 and $20\,000$ gives:

$$
\begin{aligned}
\mathbb{E}(K_{\mathcal{C}}) &\sim 0.999 \log \dim_{\mathbb{C}} \mathcal{C} + 0.581 &&\approx 0.999 \log \dim_{\mathbb{C}} \mathcal{C} + \gamma + 0.004\\
\mathbb{V}(K_{\mathcal{C}}) &\sim 0.996 \log \dim_{\mathbb{C}} \mathcal{C} - 1.043 &&\approx 0.996 \log \dim_{\mathbb{C}} \mathcal{C} + \gamma - \zeta(2) + 0.02
\end{aligned}
$$

Visually the graphs of distributions $p_{\mathcal{C}}^{Ab}(k)$ and $\frac{s(\dim \mathcal{C}, k)}{(\dim \mathcal{C})!}$ are, basically, indistinguishable for large genera.

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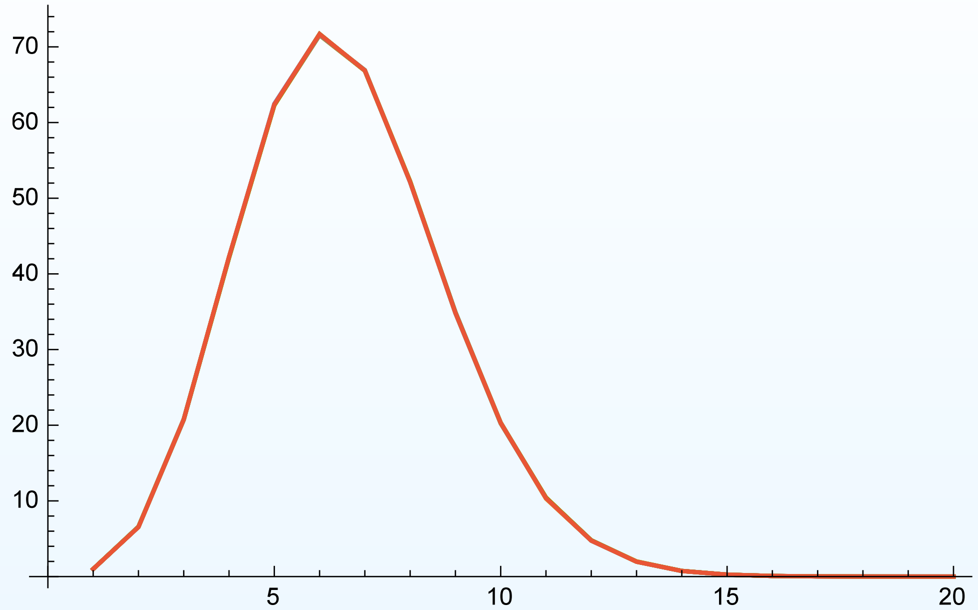
Visually the graphs of distributions $p_{\mathcal{C}}^{Ab}(k)$ and $\frac{s(\dim \mathcal{C}, k)}{(\dim \mathcal{C})!}$ are, basically, indistinguishable for large genera.

We have a rigorous (and quite involved) proof of the fact that the statistics of the number of cylinders in a random square-tiled surface in the principal stratum of quadratic differentials in genus g converges (in a very strong sense) to the statistics of the number of cycles in a random permutation of $3g-3$ elements, with respect to a very specific non-uniform measure on S_{3g-3} as $g \to +\infty$.

Conjectural large genus universality graphically



Statistics collected for the principal stratum in genus $g = 100$.

Conclusion. In order to answer Arnold's question for a concrete permutation, we need to know the Masur-Veech volume of the associated component of the stratum of Abelian differentials and, moreover, compute sophisticated contributions of k-cylinder square-tiled surfaces to this volume. Conjecturally, when the permutation becomes large enough, we need to know only the size d of this permutation (and verify that it is not hyperelliptic) and look at statistics of random permutations in S_d (for $k = 1$ we have proved this rigorously).