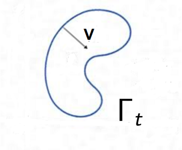
Lecture No 25 June 3, 2022 (Fri)

3.2. A quick survey of known results

3.2.1. MMC (without noise)

The hypersurface $\Gamma_t (\subset \mathbb{R}^d)$ evolves under $V = \kappa$.



- **Classical methods**
  - The problems are reduced to solve nonlinear PDEs for signed distance function, some $u(t,x)$ when Γ_t is represented as zeros of u, height function when Γ_t is represented as a graph, or $\kappa = \kappa(t,y), y \in \mathcal{M}$ under a certain diffeomorphism $\Gamma_t \ni x \leftrightarrow y = y(t,x) \in \mathcal{M}$ ($d-1$ dim manifold) and others.
  - local, global existence, long time behaviors
  - Huisken (1984): convex surfaces
  - Gage-Hamilton (1986), Grayson (1987): curves in 2D
  - Ecker-Huisken (1989): surfaces given by a graph
  - Evans-Spruck (1992): PDE for signed distance functions

- **Methods to handle solutions with singularities**
  - In general, classical solutions exist only locally in time, since singularities or topological changes may happen.
  - Brakke (1978): geometric measure theory, varifolds (non-unique)
  - Evans-Spruck (1991), Chen-Giga-Goto (1991): level set formulation and viscosity solutions (global, unique, but fattening may happen). If Γ_t is represented as $\Gamma_t = \{x \in \mathbb{R}^d; u(t,x) = 0\}$ with some $u(t,x)$, then $u(t,x)$ is a solution of

$$\frac{\partial u}{\partial t} = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

  - Note $\kappa = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$, $u \sim |\nabla u| d$ near Γ_t, $V = \frac{\partial d}{\partial t}$.
  - Books: Belletini (MMC), Lunardi (Nonlinear PDEs)

3.2.2. Stochastic motion by mean curvature (SMMC)

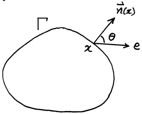
- Consider MMC perturbed by a stochastic force:

$$V = \kappa + \circ \dot{W}(t, x)$$

- $\dot{W}(t, x)$ is a white noise $\dot{W}(t)$ only in t, a colored noise or hopefully a space-time (Gaussian) white noise.
- $\circ$ means Stratonovich's sense.

- F (1999): Quasilinear SPDEs for curvature κ under the Gauss map in a convex setting in 2D, but limited to time-dependent noise only, i.e., $\dot{W}(t,x) = \dot{W}(t)$:

$$\frac{\partial \kappa}{\partial t} = \kappa^2 \frac{\partial^2 \kappa}{\partial \theta^2} + \kappa^3 + \kappa^2 \circ \dot{W}(t),$$



where $\kappa = \kappa(t,\theta), \theta \in S \simeq [0, 2\pi) \, (= \mathcal{M})$.

  - Stratonovich integral $\circ$ appears as Wong-Zakai theorem.
  - F considered a solution of corresponding martingale problem (weak solution) on $C([0,T], C^\infty(S))$ by taking a class of tame functions on $C^\infty(S)$ as test functions and proved its well-posedness.

- Lions-Souganidis (1998): stochastic viscosity solutions for the level set function $u = u(t,x)$:

$$\frac{\partial u}{\partial t} = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| \circ \dot{W}(t,x)$$

or

$$\frac{\partial u}{\partial t} = |\nabla u| \left\{ \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \circ \dot{W}(t,x) \right\}$$

- Dirr-Luckhaus-Novaga (2001), Weber (2010): rely on analysis of signed distance functions from interfaces, but limited to time-dependent noises $\dot{W}(t)$.
- von Renesse et al. (2012, 2014): interfaces described as a graph $y = h(t, x)$ (cf. Lect-23):

$$\frac{\partial h}{\partial t} = \sqrt{1+|\nabla h|^2}\,\mathrm{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right) + \sqrt{1+|\nabla h|^2} \circ \dot{W}(t)$$

  with only time-dependent noises. (If noises are acting to vertical y-direction, more general noises can be treated.)
- Physically, noises should be chosen to satisfy fluctuation-dissipation relation.
  (cf. Kawasaki-Ohta starting from TDGL eq)
- Yip (1998): Another evolution obtained by combining MMC with a stochastic flow: $V = \kappa + \langle \circ \dot{\mathbb{W}}(t, x), \mathbf{n} \rangle$.

3.3. Some further progress

3.3.1. SMMC with a direction-dependent smooth noise

(Denis-F-Yokoyama 2017; Extension of F (1999))

- In general, let us consider SMMC in 2D:

$$V = \kappa + \circ \dot{W}(t, x, \mathbf{n}(t, x)), \quad x \in \Gamma_t, \tag{1}$$

where $\mathbf{n}(t, x)$ is the inward normal vector to Γ_t at $x \in \Gamma_t$, $\dot{W}(t, x, \mathbf{n}) \equiv \dot{W}^Q(t, x, \mathbf{n})$ is a (formal time derivative of) Q-Brownian motion $W(t, x, \mathbf{n})$, i.e.,

$$W(t, x, \mathbf{n}) = \sum_{k=1}^{\infty} \psi_k(x, \mathbf{n}) B^k(t), \quad \mathbf{n} = (\cos\theta, \sin\theta) \in S \simeq [0, 2\pi),$$

with independent Brownian motions $\{B^k(t)\}_{k=1}^{\infty}$ and functions $\{\psi_k(x, \mathbf{n})\}_{k=1}^{\infty}$ decaying fast enough in k.

- At least heuristically, (1) can be derived by SIL from the stochastic Allen-Cahn equation with $\dot{W}^\varepsilon = \frac{1}{\varepsilon} \dot{W}(t, x, \mathbf{n}(t, x))$.

- If Γ_t is convex, (1) could be rewritten under the Gauss map $\theta \in S \simeq [0, 2\pi) \mapsto x(t, \theta) \in \Gamma_t$ into the SPDE for the curvature $\kappa = \kappa(t, \theta)$ at $x(t, \theta)$:

$$\frac{\partial \kappa}{\partial t} = \kappa^2 \left\{ \frac{\partial^2 \kappa}{\partial \theta^2} + \kappa + \circ \left(\frac{\partial^2}{\partial \theta^2} \dot{W}(t, x(t, \theta), \theta) + \dot{W}(t, x(t, \theta), \theta) \right) \right\},$$

  coupled with the equation for the random motion of curves $\Gamma_t = \{x(t, \theta) \in \mathbb{R}^2; \theta \in S\}$:

$$x(t, \theta) = x(0, 0) + \left(\int_0^\theta \frac{\sin \theta'}{\kappa(t, \theta')} d\theta', - \int_0^\theta \frac{\cos \theta'}{\kappa(t, \theta')} d\theta' \right) + \left(\int_0^t (\kappa + \circ \dot{\mathbb{W}})(s, 0) ds, \int_0^t \frac{\partial}{\partial \theta} (\kappa + \circ \dot{\mathbb{W}})(s, 0) ds \right),$$

  where $\dot{\mathbb{W}}(t, \theta) = \dot{W}(t, x(t, \theta), \theta)$.
- Here, $\circ$ means the product in Stratonovich's sense.

- If the noise is only direction-dependent, i.e., $\dot{W}(t,x,\theta) = \dot{W}(t,\theta)$, then the SPDE for κ has a simple form:

$$\frac{\partial \kappa}{\partial t} = \kappa^2 \left\{ \frac{\partial^2 \kappa}{\partial \theta^2} + \kappa + \circ \dot{\mathbf{W}}(t,\theta) \right\},$$

  and we don't need to couple with the equation for $x(t,\theta)$.
- Here, $\dot{\mathbf{W}}(t,\theta) = \frac{\partial^2}{\partial \theta^2} \dot{W}(t,\theta) + \dot{W}(t,\theta) = \sum_k \varphi_k(\theta) B^k(t)$ with $\varphi_k = \partial_\theta^2 \psi_k + \psi_k$.
- This is equivalent to the SPDE of Itô form:

$$\frac{\partial \kappa}{\partial t} = \kappa^2 \frac{\partial^2 \kappa}{\partial \theta^2} + \kappa^3 (1 + \varphi(\theta)) + \kappa^2 \dot{\mathbf{W}}(t,\theta),$$

  where $\varphi(\theta) = \sum_k \varphi_k(\theta)^2 (= \mathrm{Tr}\, q)$.
- We assume the covariance of Q-BM $\mathbf{W}(t,\theta)$ satisfies

$$q(\theta_1,\theta_2) := \sum_{k=1}^{\infty} \varphi_k(\theta_1) \varphi_k(\theta_2) \in C^\infty(S \times S).$$

- Pathwise uniqueness follows from an energy inequality.
- Existence: If the noise term is smooth also in t, one can apply the PDE result due to Giga-Mizoguchi. Then, one can pass to the limit by showing the tightness of an approximating sequence. This shows the existence of a local (in time) solution in law sense (weak solution).
- However, due to Yamada-Watanabe type result (e.g., see Ondreját), the pathwise uniqueness with the existence of a local weak solution implies the existence of a local strong solution.

3.3.2. Volume preserving MMC with noise

- Consider
$$V = \kappa - \mathrm{av}_{\Gamma_t}(\kappa) + c\dot{W}_t, \tag{2}$$
where $\mathrm{av}_\Gamma(\kappa) = \frac{1}{|\Gamma|}\int_\Gamma \kappa$ is the average of κ over Γ.
- If $c = 0$, the volume inside Γ_t is preserved in t.
- F-Yokoyama 2019 (cf., Section 4) derived (2) with multiplicative noise $\frac{c}{|\Gamma_t|} \circ \dot{W}_t$ under SIL from stochastic volume-preserving Allen-Cahn equation, but only for Γ_t being 2D convex curve.

C: Sharp interface limit

4.1. Sharp interface limit (SIL) without noise

4.2. Sharp interface limit with noise

4.2.1. $d = 1$

4.2.2. $d \geq 2$

4.2.3. Stochastic mass-conserving Allen-Cahn equation

(1) Equation

(2) Sharp interface limit (F-Yokoyama, Ann Probab 2019)

- Results
- Heuristic argument (difference in scaling of noise)

(3) Asymptotic expansion method

- Derivation of the evolutional law of limit hypersurface
- Schauder estimate for parabolic operator with diverging coefficients

(4) Limit Stochastic PDE — 2D, convex curve case

(5) Summary

4.2.4. The case with boundary condition (Lee)

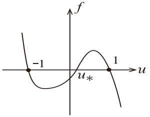
4.1. Sharp interface limit (SIL) without noise

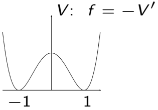
- Recall Allen-Cahn equation:

$$\frac{\partial u}{\partial t} = \Delta u + \frac{1}{\varepsilon^2} f(u), \quad t > 0,\ x \in D \subset \mathbb{R}^d,$$

  usually with Neumann boundary condition at ∂D, where $\varepsilon > 0$ is a small parameter.

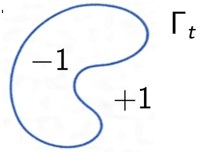
- f is bistable with stable points ± 1 and unstable point $u_* \in (-1, 1)$ i.e., $f\ (= -V') \in C^\infty(\mathbb{R})$ s.t. $f(\pm 1) = f(u_*) = 0,\ \ f'(\pm 1) < 0,\ f'(u_*) > 0.$





- Example: $f(u) = u - u^3$.

- We expect to have $\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(t,x) = +1$ or -1. The problem is to find the evolutional law of the interface Γ_t separating two phases ± 1 under proper time scale.



- The time scale highly depends on

$$A(f) := \int_{-1}^{1} f(u)\,du \; \big(= V(-1) - V(1) \big),$$

where V is the corresponding potential s.t. $f = -V'$.

(a) The case $A(f) \neq 0$: The proper time scale is $O(\varepsilon)$, i.e., for the solution u^ε of AC eq, we have

$$\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon t, x) \underset{\varepsilon \downarrow 0}{\longrightarrow} \chi_{\Gamma_t}(x) := \begin{cases} +1, & \text{outside of } \Gamma_t, \\ -1, & \text{inside of } \Gamma_t, \end{cases}$$

The hyperplane Γ_t in D evolves according to the Huygens' principle: waves with speed $c(f)$ are created from each point of Γ_t to all outward directions, and Γ_t is determined as the envelope of the wave fronts (Gärtner 1983).

(b) The case $A(f) = 0, d \geq 2$: Since $c(f) = 0$, the wave does not move at the time scale $O(\varepsilon)$. The proper time scale is $O(1)$, i.e.,

$$u^{\varepsilon}(t,x) \longrightarrow \chi_{\Gamma_t}(x) \qquad (\varepsilon \downarrow 0)$$

and Γ_t moves according to motion by mean curvature: $V = \kappa$. The heuristic argument to derive MMC was given in Sect 3.1.2.

(c) The case $A(f) = 0, d = 1$: This is a plane wave so that the proper time scale is much longer than $O(1)$. In fact, Carr-Pego 1989 showed that the order is $O(\exp C\varepsilon^{-1})$.

- In the following, we consider the case (b) and always assume no flux (balanced) condition: $A(f) = 0$

Derivation of MMC is discussed by

- de Mottoni-Schatzman (1995), X. Chen (1992): asymptotic expansions
- Evans-Soner-Souganidis (1992), Soner (1997), Barles-Souganidis (1998): Extension of these results in viscosity setting
- Ilmanen (1993): Extension by varifold setting

- **Extensions**
  - Alfaro-Garcke-Hilhorst-Matano-Schatzle (2010): SIL leading to anisotropic MMC ($\Delta \to$nonlinear Laplacian: $\partial_t u = \operatorname{div} a(\nabla u)) + \frac{1}{\varepsilon^2} f(u)$.)
    - F-Spohn (1997): derivation of anisotropic MMC from microscopic model
  - Cahn-Hilliard equation (to Hele-Shaw problem; cf. stochastic case, Rockner-Yang-Zhu, 2021)

$$\frac{\partial u^\varepsilon}{\partial t} = -\varepsilon \Delta^2 u^\varepsilon + \frac{1}{\varepsilon} \Delta \{ f(u^\varepsilon) \}$$

    and phase-field model
  - Chen-Hilhorst-Logak (2010): Volume preserving MMC under SIL for mass conserving Allen-Cahn equation:

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2} \left(f(u^\varepsilon) - \fint_D f(u^\varepsilon) \right) \underset{\varepsilon \downarrow 0}{\longrightarrow} V = \kappa - \fint_{\Gamma_t} \kappa$$

    No comparison argument $\quad (\fint_D := \frac{1}{|D|} \int_D, \ \fint_\Gamma := \frac{1}{|\Gamma|} \int_\Gamma)$

4.2. Sharp interface limit (SIL) with noise

- Kawasaki-Ohta (1982): Derivation of SMMC due to sharp interface limit for the stochastic Allen-Cahn (SAC) equation (TDGL equation):

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u) + \dot{W}^\varepsilon(t,x), \quad x \in \mathbb{R}^d \text{ (or } D) \quad (3)$$

- $\dot{W}^\varepsilon(t,x)$ is a space-time noise (not necessarily the space-time white noise) depending on a small parameter $\varepsilon > 0$.
- We only consider the cases (b) and (c) with noises, so that our assumptions are: f is bistable and $A(f) = 0$.

- We also assume a technical condition:

$$\exists C, p > 0 \text{ s.t. } |f(u)| \leq C(1 + |u|^p), \quad \sup_u f'(u) < \infty.$$

- The second is the monotonicity condition which includes cubic $f(u) = u - u^3$.
- Recall that the solution is defined in a sense of generalized functions or equivalently as a mild solution.
- With the space-time Gaussian white noise,

$$u(t, x) \in C^{\frac{2-d}{4}-, \frac{2-d}{2}-} a.s.$$

- Well-posed only when $d = 1$.

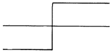
4.2.1. $d = 1$: $D = \mathbb{R}$, $\dot{W}^\varepsilon(t,x) = \varepsilon^\gamma a(x)\dot{W}(t,x)$

- F (1994, 95, 97): $\gamma > \frac{19}{2}$ (smallness of noise), $a \in C_0^2(\mathbb{R})$ is an intensity of the noise (spatial cut-off), $\dot{W}(t,x)$ is a space-time Gaussian white noise (only 1D) and assume f is symmetric (i.e., $f(u) = -f(-u)$, in particular $A(f) = 0$).
- The SPDE

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u) + \varepsilon^\gamma a(x)\dot{W}(t,x), \quad x \in \mathbb{R},$$

has a unique solution s.t. $u^\varepsilon(t,x) \in C^{\frac{1}{4}-,\frac{1}{2}-}((0,\infty)\times\mathbb{R})$

- If $u^\varepsilon(0,x) \to \chi_{\xi_0}(x)$, then we have



$$u^\varepsilon(t,x) := u(\varepsilon^{-2\gamma-1}t, x) \longrightarrow \chi_{\xi_t}(x) = 1_{(\xi_t,\infty)} - 1_{(-\infty,\xi_t)}$$

- More precisely,

[Theorem 1] (F, PTRF 1995, Proc. Taniguchi sympo 1997)
If the initial value has the form $u^\varepsilon(0,x) = m((x-\xi)/\varepsilon)$ and the reaction term has the symmetry $f(u) = -f(-u)$, then for $\gamma > \frac{19}{2}$,

$$\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon^{-2\gamma-1}t, x) \Longrightarrow \chi_{\xi_t}(x) \quad (\varepsilon \downarrow 0),$$

where $\chi_\xi(x) = 1(x > \xi)$, $\chi_\xi(x) = -1(x < \xi)$. The phase separation point ξ_t moves according to the following SDE:

$$d\xi_t = \alpha_1 a(\xi_t) dB_t + \alpha_2 a(\xi_t) a'(\xi_t) dt, \quad \xi_0 = \xi, \tag{4}$$

where B_t is a 1D Brown motion, $\alpha_1 = ||m'||^{-1}_{L^2(\mathbb{R})}$,

$$\alpha_2 = -||m'||^{-2}_{L^2(\mathbb{R})} \int_0^\infty dt \int_{\mathbb{R}^2} xp(t,x,y)^2 f''(m(y)) m'(y) dxdy,$$

and $p(t,x,y)$ is a fundamental solution of the linearized operator $\partial/\partial t - \{\partial^2/\partial y^2 + f'(m(y))\}$. □

- This theorem shows the diffusion coefficient (mobility) α_1^2 is given by the inverse of the surface tension $||m'||_{L^2(\mathbb{R})}^{-2}$ and this coincides with the conjecture made by Kawasaki-Ohta 1982 and Spohn 1993.
- Numerical simulation (by Y. Otobe):
  (1) $\varepsilon = 0.01, \gamma = 0.25$: strong force to ± 1 compared to (2), BM
  (2) $\varepsilon = 0.1, \gamma = 0.25$: BM
  (3) $\varepsilon = 0.1, \gamma = 0.375$: bit small fluctuation compared to (2), BM
  (4) $\varepsilon = 0.1, \gamma = 2.0$: very small but some fluctuation is observed, time is too short to observe BM

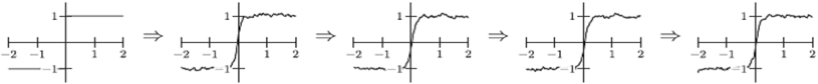


図 5.12　波の動き (時空ホワイトノイズ, $\varepsilon = 0.01$, $\gamma = 0.25$)

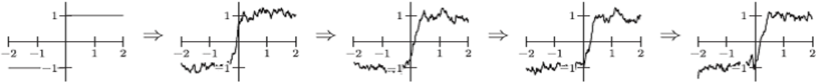


図 5.13　波の動き (時空ホワイトノイズ, $\varepsilon = 0.01$, $\gamma = 0.01$)

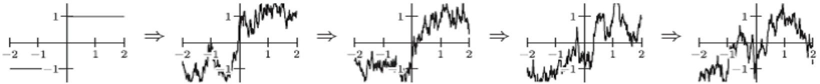


図 5.14　波の動き (時空ホワイトノイズ, $\varepsilon = 0.2$, $\gamma = 0.0001$)

ε^2 is denoted by ε so that $\varepsilon = 0.01$ in figure means $\varepsilon = 0.1$ in our setting

Noise is $\dot{W}(t)$

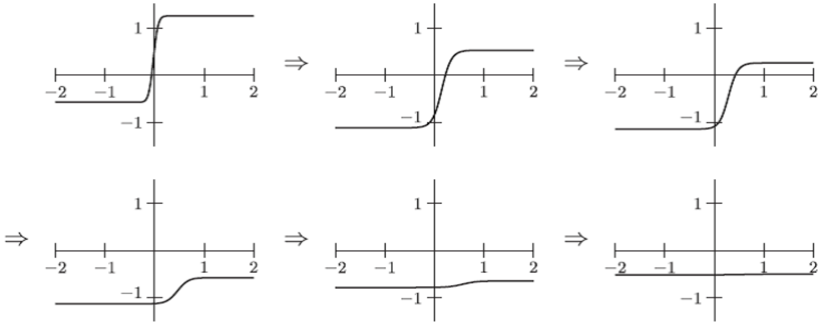


図 5. 11　波の動き（時間のみのノイズ，$\varepsilon = 0.01$, $\gamma = 0.25$）

ε^2 is denoted by ε so that $\varepsilon = 0.01$ in these figures means $\varepsilon = 0.1$ in our setting

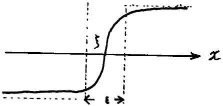
Intuitive reason for the time change $\varepsilon^{-2\gamma-1}$

- The time change $\varepsilon^{-2\gamma-1}$ is very different from the case without noise, recall Carr-Pego.
- The intuitive reason that this is the proper time scale is explained as follows: $\bar{u} = \bar{u}^{\varepsilon}$ satisfies (in law):

$$\frac{\partial \bar{u}}{\partial t} = \varepsilon^{-2\gamma-1}\left\{\Delta \bar{u} + \frac{1}{\varepsilon^2} f(\bar{u})\right\} + \left(\varepsilon^{-2\gamma-1}\right)^{1/2} \cdot \varepsilon^{\gamma} a(x) \dot{W}(t,x).$$

- Noise term is $a(x)\varepsilon^{-1/2}\dot{W}(t,x)$.
- The strong drift $\varepsilon^{-2\gamma-1}$ pushes $\bar{u}$ to the neighborhood of

$$M^{\varepsilon} := \left\{\bar{u};\ \Delta \bar{u} + \frac{1}{\varepsilon^2} f(\bar{u}) = 0,\ \bar{u}(\pm\infty) = \pm 1\right\}$$
$$= \{m((x-\xi)/\varepsilon);\ \xi \in \mathbb{R}\}$$



so that $\bar{u}^{\varepsilon}(t,x) \sim m((x-\xi_t)/\varepsilon)$.

- In particular, the width of the interface is $O(\varepsilon)$.
- The contribution of the noise $\dot{W}(t,x)$ comes only from this region, therefore its order is $O(\varepsilon^{1/2})$ by self-similarity.
- This balances with the factor $\varepsilon^{-1/2}$ in front of the noise.

- This explains the properness of the time change $\varepsilon^{-2\gamma-1}$.