Applications of Calabi-Yau periods to Scattering Amplitudes

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Application of Calabi-Yau periods to scattering Amplitudes

3 x 95 min

I| Definition, Properties & evaluation of CY-Periods

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I.1 Calabi-Yau manifolds X

Def. A CY n-fold is a n-complex dimensional Kähler manifold with SU(n) holonomy

- It is a complex manifold : • One has holomorphic transition function between its charts in $\mathbb{C}^n$ (parametrised by x_k $k=1..n$)
  • ∃ an integrable almost complex structure
- It is Kähler manifold : • ∃ Kähler form ω of Hodge type (1,1) so that $\omega^n \sim$ volume form of X
- SU(n) holomy implies that local a (n,0)-form $f(x)dx_1\wedge\ldots\wedge dx_n$ extends to to global no-where vanishing (n,0) form Ω
- This implies that the canonical bundle K_X is trivial $K_X=0$ or equivalently that the first Chern class $c_1(T_X)=0$.

- Calabi conjectured and Yau proved that in a given Kähler class there exist a unique Ricci flat metric g, i.e. $R_{i\bar{j}}(g) = 0$.

Examples: Elliptic curves are CY 1-folds

Specific Example: a) Legendre family of ell curves

Zagier: Arithmetic & topology of Diff eq

$$E_t : wy^2 = x(x-1\cdot w)(x-tw) \quad \in \mathbb{P}^2 \ (w:x:y) \quad (1)$$

Family of elliptic curves

complex structure parameter

$$\Omega_t = \frac{dx}{y(t)} \qquad \omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$$

Hodge decomposition (p,q): $(1,1)$ ω; $(1,0)$ Ω; $(0,1)$ $\bar{\Omega}$; $(0,0)$ $\mathcal{O}_{E_t}$

b) Quintic in $\mathbb{P}^4$

X_t Family of CY manifolds

$$0 = P = \underbrace{X_1^5 + \ldots + X_5^5 + 5\overset{a_0}{t^{-1/5}} X_1 \ldots X_5}_{\text{in } \mathbb{P}^4}, + 100 \text{ ineq. def } (2)$$

one complex structure paramer

$$0 \to TX \to T\mathbb{P}^4 \to N \to 0$$

$\dim(H^{n-1,n}) = \dim(\mathcal{M}(x))$

$$H^{n-1,1}(X) \overset{\lrcorner\Omega}{=} H^1(X, TM)$$

complex structure def unobstructed Tian Todorov

$$ch(TX) = \frac{(1+H)^5}{1+5H} = 1 + \underset{c_1=0}{(5-5)H} + \underset{c_2}{10H^2} - \underset{c_3}{40H^3}$$

$$\int_{TX} c_X = \int_{\mathbb{P}^4} c_X \, 5H$$

$$H^4 = 1 \qquad [c \cdot H] = \int_{TX} c_2 \cdot H = 50$$

101 complex def

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

no global symm inver

Euler number $\quad e = \int_{TX} c_3 = -200$

Griffith Res form $\quad P=0$

$$\Omega = \frac{1}{2\pi i} \oint_\gamma a_0 \frac{\mu}{P} \quad (*) \qquad \mu = \sum_{k=1}^{5} x_k \, dx_1 \ldots \widehat{dx_k} \ldots dx_5$$

Exercise: a) $c_1(E) = 0$, b) GR-form yields $\Omega = \frac{1}{2}\frac{dx}{y}$ of quartic in $\mathbb{P}^3$ $\quad e = 24$ $\quad 1 \quad (1,19) \quad 1$

$\omega \nearrow \quad 1 \quad \nwarrow$ def

polarized 19-dim K3 family

$b_2 = 22 \qquad \mathrm{sign}(H_2(K3)) = (19,3)$

Intersection form on $H_3(K3,\mathbb{Z})$: $\Sigma_{K3} = H^{\otimes 3} \oplus E_8 \oplus E_8$, $\quad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Reading:

I.2) Periods on Calabi–Yau manifolds

$$\Pi_{ji}(\underline{t}) = \int_{\Gamma_i} \gamma^j(t) \qquad (3) \qquad \text{Period matrix}$$

Γ_i basis of Homology $H_n(X,\mathbb{Z})$

γ^j basis of Cohomology $H^n(X,*)$

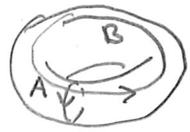
Periods: $\quad \Pi : H_n(X_n,\mathbb{Z}) \times H^n(X_n,\mathbb{C}) \to \mathbb{C} \qquad (4)$

Additional important data: Intersection form

$$\Sigma : H_n(X,\mathbb{Z}) \times H_n(X,\mathbb{Z}) \to \mathbb{Z} \qquad (5)$$

$$\Sigma = \begin{cases} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}_{b_n \times b_n} & \text{symplectic if } n \text{ odd} \\ \text{even (e.g. } \Sigma_{K3}) & \text{if } n \text{ even} \end{cases} \qquad (6)$$

Example: $n = 1 \qquad \Gamma = \langle A, B \rangle$



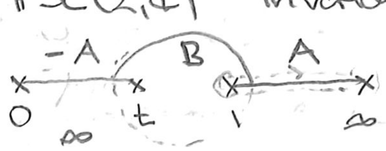
$A \cap B = 1 = -B \cap A$

$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(1) branched double cover of $\mathbb{P}^1$

$x \to \dfrac{ax+b}{cx+d}$ $\quad$ $PSL(2,\mathbb{C})$ invariant on $\mathbb{P}^1$ allows to put 3 points to $0,1,\infty$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$



$\binom{2n}{n}^2 = \binom{-1/2}{n}^2 2^{4n}$

$$\Pi_{11} = \int_A \Omega = 2\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}} = 2\pi \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 t^n = {}_2F_1\left(\tfrac{1}{2}\,\tfrac{1}{2};1,t\right) \qquad (7)$$

Exercise: Calculate

(8) $\Pi_{12} = \int_B \Omega = 2\int_t^1 \Omega \underset{\mathrm{Ex}}{=} \frac{2}{i} \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 \left(\log\left(\frac{t}{16}\right) + 4(H_{2n} - H_n)\right) t^n$

$H_k = \sum_{\ell=1}^{k} \frac{1}{\ell}$ Harmonic number

(7) (8) are Elliptic integrals

Observations: Maximal Cut Feynman graphs in critical dimension:

1 loop	rational Functions
2 loop	elliptic "
3 "	K3-periods ...

Actual Feynman graphs (Non maximal cut) : multPolylogarithms, which are iterated integrals of rat functions:

$\frac{x}{1-x} = \sum_{k=1}^{\infty} x^k = \mathrm{Li}_0(x)$; $\int_0^x \frac{dx}{x} \mathrm{Li}_n = \mathrm{Li}_{n+1}(x)$ $\quad \mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$

Main theme of the lecture: We need to generalize this and develop a theory of iterated CY periods to understand higher loop perturbation theory in QFT and many other applications (elliptic Poly logs already developped)

I.3) Picard Fuchs operators & Monodromies

Notation.
$\partial := \partial_t$
$\theta = t\partial_t$

A trivial observation

Δ discriminant
$\shortparallel$

$$dx = -2\, d\left(\frac{x^{1/2}(x-1)^{1/2}}{(x-t)^{3/2}}\right) = \left(4t(t-1)\partial^2 + 4(2t-1)\partial + 1\right)\Omega_t$$
$$= \left(\theta^2 - t\left(\theta + 1/2\right)^2\right)\Omega_t = \mathcal{L}^{(2)}\Omega_t$$

As Π_{ij} are integrals over closed cycle we observe that

(9) $\mathcal{L}^{(2)}\, \Pi_{ij} = 0$ $\quad \mathcal{L}^{(4)}$: Picard Fuchs operator

for Legendre Curve

(5) Of course we may check that (ω) annihilates (7)+(8), but the latter converge only for $|t|<1$. (ω) gives some control on the analytic properties of $\Pi_{ij}(t)$ over $\underbrace{\mathcal{M}_{cs}(E_t)}$ $= \mathbb{P}^1 \setminus \{0, 1, \infty\}$

Complex moduli space

$\delta = t, \quad \delta = 1-t, \quad \delta = \frac{1}{t}$ local coordinates

local exponents

with Solution ansatz $\quad f(t) = \delta^{\alpha} \sum_{n=0}^{\infty} c_n \delta^n$

The Riemann symbol samarizes the info about the local exp.

$$\mathcal{R} = \left(\begin{array}{ccc} z=0, & 1, & \infty \\ \hline 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{array} \right)$$

and gives the local monodromy.

local exponents

However (ω) gives solution over $\mathbb{C}$ not actual geometric periods, which are also the Feynman Integrals

A global modromy analysis allows to reconstruct the geometric periods.

Example: Using analytic continuation we can chose global solution so that the Monodromies are

$$M_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$$

Relation $M_0 M_1 M_\infty = \mathbb{1}$ $\qquad M_0, M_1$ generate

$$\Gamma_0(2) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; M \bmod 2 = \mathbb{1} \right\}$$

Generally we have: $\quad M_x^T \Sigma_1 M_x = \Sigma_1$

- M_x generates $\Gamma_x \subset O(\Sigma_1, \mathbb{Z}) = \Gamma_{\Sigma_1}$
- If Γ_x acts irreducible, the PF operator and the global monodromy determines a geometric basis up to Γ_Σ transformations.

Since on a general CY integral especially the one of type (B) is hard to compute one puts the Picard-Fuchs operator (ideal) or Gauss Manin connection, monodromy in the center of the theory.

- Easy residue integrals are notable exeptions:

Example: Consider the $T^{n=3}$ integral of (2) (symmetric) (*)

$$\Pi_{11} = \left(\frac{1}{2\pi i}\oint\right)^4 \frac{a_0\mu}{P} = \left(\frac{1}{2\pi i}\oint\right)^4 \frac{\mu}{x_1x_2x_3x_4x_5\left(1-\underbrace{\frac{1}{5}t^{-1/5}\left(\frac{x_1^4}{x_2x_3x_4x_5}+\ldots\right)}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5}\left(\frac{t}{5^5}\right)^n$$

Constant term in geometric series

We have $\mathcal{L}^{(4)} = \theta^4 - t\prod_{k=1}^{4}\left(\theta+\frac{k}{5}\right)$ $\quad \mathcal{L}^{(4)}\Pi_{1*}=0$

orbifold

$$R = \begin{pmatrix} z=0 & 1 & \infty \\ 0 & 0 & 1/5 \\ 0 & 1 & \vdots \\ 0 & 1 & \vdots \\ 0 & 2 & 4/5 \end{pmatrix}$$

point of maximal unipotent monodromy $(M_0-1)^{k+1} = 0 \quad k \geq n$

conifold point

Generally $\mathcal{L}^{(n)}$ can be derived using partial integration (Griffith - Dwork reduction)

$$\frac{a_0 + Q(x,a)\,\partial_k P}{P^{r+1}}\,\mu = \frac{a_0\,\partial_k Q(x,a)}{P^r}\,\mu, \qquad (11)$$

with $Q(x,a)$ polynomal with right scaling under $x_i \to \lambda^{w_i} x_i$, so that (11) well defined

and Gröbner basis reduction. $P = dP = 0$ generically singular Pierre Lairez & Pierre Vanhove 2209.10962

- Reduction of Gelfand-Kapranov-Zelevinski System for Komplete Intersection in toric ambient spaces Hosono, Theisen, Yau AK(93) in Review

I.4) The Riemann bilinear equations & Calabi-Yau operators

$$\underline{\Pi} = \Pi_{1*}$$

$$0 \leq e^{-K} = i^{n^2} \int \Omega \wedge \overline{\Omega} = i^{n^2} \underline{\Pi}^{+} \Sigma^{\prime} \underline{\Pi} \qquad (12)$$

$$\int \Omega\, \partial^{K}_{I_K} \Omega = \underline{\Pi}^{T} \Sigma^{\prime}\, \partial^{K}_{I_K} \underline{\Pi} = \begin{cases} 0 & K<n \\ C_{I_K}[t] & K=n \end{cases} \qquad (13)$$

$\uparrow$ rational function in t

ad (12)

- K real function Kähler potential on $M_{cs}(X_t)$ with

<u>Weil Petersson Metrik</u> $\quad G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K \qquad (14)$

- $\Omega_t \wedge \overline{\Omega}_t = \varphi(t) \underbrace{\frac{\omega^n}{n!}}_{\text{Volume form}}$ Gross Huybrecht Joyce "CY M. and related geometries"

calibration

ad (13) $\quad F^p H^n = \bigoplus_{l>p} H^{l,n-l} \qquad F^3 H^3 = H^{30}$ $\qquad (14)$

$F^2 H^3 = H^{30} \oplus H^{21}$

$\downarrow$

holomorphic free locally constant sheaves over $M_{cs}(X)$

$$\mathcal{F}^0 \supset \dots \supset \mathcal{F}^n \supset \mathcal{F}^{n+1} = 0$$

$\quad \|$

$R^n \pi_* \mathbb{C} \otimes \mathcal{O}_{M_{cs}}$

$\Rightarrow$ Flat Gauss Manin Connection $\quad \nabla : \mathcal{F}^0 \mapsto \mathcal{F}^0 \otimes \Omega^1_{M_{cs}} \qquad (15)$

with $\quad \nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_{M_{cs}} \qquad (16)$

Local coordinates on M_{cs} $\quad \underline{t} \qquad \partial^K_{I_K} = \partial_{t_{i_1}} \dots \partial_{t_{i_K}}$

$$\partial^K_{I_K} \mathcal{F}^p \subset \mathcal{F}^{n-K}$$

- Consideration of type yields 0 case in (13)

Flatness of Gauss Manin connection

$$\nabla_t = \partial_t - M_t(t), \quad \nabla_t \Pi = 0 \qquad (14)$$

is equivalent first order formulation to

$$\mathcal{L}^{(n_K)}_K(t)\, \Pi = 0$$

$\{\mathcal{L}^{(n_K)}_K(t)\}$ generate complete differential ideal I_D

<u>Example</u>: $\dim(\mathcal{M}_{cs}) = 1$

$$\mathcal{L}^{(n+1)} = \partial^{n+1} + \sum_{i=0}^{n} a_i(t)\partial^i$$

$C = C_{\underbrace{1\ldots1}_{n}}$ $(13) \Rightarrow \frac{\partial_t C(t)}{C(t)} = \frac{2}{n+1} a_n(t)$ and $\mathcal{L}^{(n+1)}$

essentially self adjoint. Def $\mathcal{L}^{*(n+1)} = \sum_{i=0}^{n+1} (-\partial)^i a_i(t)$

$\hookrightarrow$ $\mathcal{L}^{*(n+1)} C(t) = (-1)^{n+1} C(t) \mathcal{L}^{(n+1)}$ $\qquad (15)$

- In general I_D determines $C_{I_n}(t)$

- $S^K = \frac{\Pi_K}{\Pi_0}$ $\quad K = 1, \ldots, h_{n-1,1}$ $\quad$ Inhomogenous coordinates on $\mathcal{M}_{cs}(X)$

Example: $n=3$ $(13) \Rightarrow$

$$\Pi = \Pi_0 \begin{pmatrix} 1 \\ S^K \\ \partial_K \mathcal{F} = \mathcal{F}_K \\ 2\mathcal{F} - S^K \partial_K \mathcal{F} \end{pmatrix} \qquad \mathcal{F}(S) \text{ pre potential} \qquad (16)$$

Special Kähler geometry (SKG)

$$\left\{ \begin{array}{l} \partial_i \partial_j \partial_K \mathcal{F} = C_{ijK} \in \mathcal{L}^2 \otimes \mathrm{Sym}^3(T^*_{cs}) \\ \qquad \Omega \in \mathcal{L} \leftarrow \text{Kähler line bundle} \\ (12) \Rightarrow \; e^{-K} = i\left((S^K - \bar{S}^{\bar{K}})(\mathcal{F}_K + \bar{\mathcal{F}}_{\bar{K}}) + 2(\mathcal{F} - \bar{\mathcal{F}})\right) \end{array} \right.$$

- $W = \begin{pmatrix} \Pi_0 & \ldots & \Pi_n \\ \partial^n_t \Pi_0 & & \partial^n_t \Pi_n \end{pmatrix}$ $\quad (13)$ implies $\quad W^T \Sigma W = Z(t)$ $\quad$ rational function

There fore we can write

$$W^{-1} = Z^{-1} W^T \Sigma \qquad (16)$$

I.5 Griffith Normal function, inhomogeneous Picard Fuchs equations & Iterated integrals of CY Periods

Böhnisch, Duhr, Nega, Fischbach, AK 2108.05310

With hind sight to applications we note that the non-maximal cut (actual Feynman) integral is a chain integral

$C \in H_*^{rel}(X, L)$

$$\underset{\substack{\uparrow}}{I_{GN}} = \int_C \Omega \quad (18) \qquad \text{with } \partial C = \cup B_k \neq 0$$

Griffith Normal function, which fullfilles the following inhomogeneous differential equations

$$\underset{\substack{\uparrow \\ \text{non PF} \\ \text{operator}}}{\mathcal{L}^{(n)}(t)} \; \underset{\substack{\uparrow \\ \text{special} \\ \text{solution}}}{I_{GN}} = b(t) \qquad (19)$$

Application of variation of constand method yields

periods: basis of solution to $\mathcal{L}^{(n)} \Pi = 0$

$$\text{Anstatz } I_{GN} = \sum_{k=0}^{\overbrace{n_{n-1}}^{m}} C_k(t) \, \Pi_k \qquad \text{One has } W \underline{C}' = \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}}_{= B} \Big\}^{m} \qquad \underline{C}' = W^{-1} B$$

linear in periods

$$I_{GN} \overset{(18)}{=} \sum_{k=0}^{m-1} \Pi_k \underbrace{\int_0^t \left[(Z^{-1} W^T Z_1) \cdot B \right]_k}_{\text{iterated Calabi-Yau Periods}} \qquad (19)$$

↑ rational function

II Basic applications of CY periods to Feynman Integrals

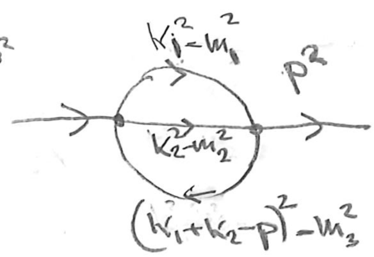
II.1 Feynman integrals: We focus on the special Functions that occur in the evaluation. For this it is sufficient to reduce spin dependence and focus on scalar integrals for a given graph G. The typical integral is of the form

$$I_\nu(\underline{m},\underline{p}) = \int \prod_{r=1}^{\ell} \frac{d^{D_\varepsilon} k_r}{i\pi^{D/2}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_i}} \qquad (20)$$

- $D_\varepsilon = d - \varepsilon$ (ε: dimensional regularisation parameter; d: critical space time dimension)
- $D_j = q_j^2 - m_j^2 + i0$ Scalar Feynman Propagator (masses $m_j^2 \subset \mathbb{R}_+$ masses)
- k_r^2 Loop momenta ℓ- loop order

  q_j determined from external momenta & loop momenta by momentum conservation at each Vertex

Example: 2 loop Banana graph



- $\nu_i \in \mathbb{Z}$ Propagator powers are at this point $\nu_i = 1 \ \forall i$
- Lorentz invariance implies that $I(\underline{m},\underline{p})$ depends only on Lorentz invariant scalar products

i.e. $x_k = p_i \cdot p_j$, $i,j = 1,\dots,\#\text{ext}$, $x_k = m_k^2$. Dimensional analysis implies, that $I(\underline{m},\underline{p})$ depends only on dimensionless ratios

$$t_k = \frac{x_k}{x_{r+1}} \qquad k = 1,\dots,r \qquad (21)$$

- Note: If # Propagators is not $r+1$ one has additional Lorentz inv. in numerator of (20)

II.2 Special geometric Representations of Feynman Integrals

The Symanzik representation: (using Feynman parameters $I_G(t)$ inv $(x_1 \dots x_n) \sim (\lambda x_1 \dots \lambda x_n)$)

$$I_G(\underline{t}) \sim \int_{\sigma_{n-1}} \prod_{i=1}^{n} x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{d}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1} \qquad (22)$$

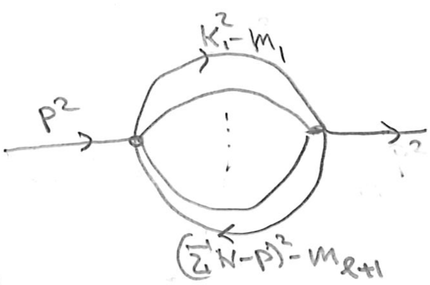
measure on $\mathbb{P}^{n-1}$ see (*)

$\mathcal{U}(x)$, $\mathcal{F}(x,\underline{t})$ } Graph- or Kirchhoff polynomials that depend on Combinatoric of Graph. Bogner & Weinzierl 1002.3458

$$\omega = \sum_{i=1}^{\#\text{ed}} \nu_i - \ell\, d/2$$

chain: $\sigma_{n-1} = \{ [x_1 : \dots : x_n] \subset \mathbb{P}^{n-1} \mid x_i > \mathbb{R}_> \text{ for } 1 \le i \le n \}$

II.3 Example: ℓ-loop Banana Integral in $d=2$ dim



$t_k = \frac{m_k^2}{p^2}$

$\nu_i = 1$

$\omega = \ell + 1 - \ell = 1$

$\mathcal{F} = \left(\prod x_i \cdot p^2\right) \tilde{\mathcal{F}}$

$$\tilde{\mathcal{F}} = \left(1 - \left(\sum_{i=1}^{\ell+1} \frac{1}{x_i}\right)\left(\sum_{i=1}^{\ell+1} x_i t_i\right)\right)$$

$$I_{Ban} \sim \int_{\sigma_{n-1}} \prod_{i=1}^{l+1} x_i \frac{1}{\tilde{\mathcal{F}}} \mu_{n-1} \qquad (23)$$

maxical cut integral is obtained by replacing $\sigma_{n-1} \to T^{n-1}$ integral

Comparison with (*) after (2) shows that

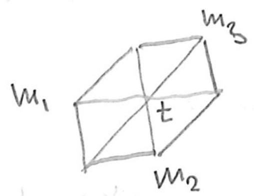
$$I_{Ban} = \int_{T^{n-1}} \frac{1}{\tilde{\mathcal{F}}} \mu_{n-1} = \int_{T^{n-2}} \Omega_{Ban} \quad \text{Period integral!}$$

<u>Exercise</u>: a) Perform residuum integral to show that for $t_1 = \ldots = t_5 = t$

$$\mathcal{L}^{(4)} = (1-25t)(1-16t)(1-9t)(1-t)\theta^4 + (4-70t+450t^3)\theta^3$$
$$-(6-63t+26t^2-225t^3)\theta^2 + (4-28s)\theta - 1 + 5s \qquad (22)$$

annihilates the $l=5$ <u>equal mass</u> Banana integral!
BFKNS Bönisch, Fischbach, Klemm, Nega, Safari 2008.10574

b) The Newton polyhedron Δ $\tilde{\mathcal{F}}$ is a reflexive polyhedron, e.g.



for $l=3$. Hence $X_{Ban} = \{\tilde{\mathcal{F}} = 0 \subset \mathbb{P}_{\Delta^*}\}$

is a mirror pair of Calabi-Yau $(l-1)$-folds via the Batyrev Construction. J. Alg Geom 3 (1994) 493

For $l > 3$ the physical parameters $t_1, \ldots, t_{l+1}$ are only a small subspace of $\mathcal{M}_{cs}(X_{Ban})$ and the GKZ-System associated to $\mathbb{P}_{\Delta^*}$ is highly redundant.

One can show that there is a transition of X_{Ban} to the complete intersection

(24) $$\hat{X}_{Ban} = \begin{pmatrix} \mathbb{P}^1_{(1)} & 1 & 1 \\ \vdots & \vdots & \vdots \\ \mathbb{P}^1_{(l+1)} & 1 & 1 \end{pmatrix} \quad \begin{matrix} S^1 \\ \vdots \\ S^{l+1} \end{matrix}$$ are Kähler parameter of its mirror

Using Hosono, Theisen Yau AK 9406055. The associated GKZ System with

$$\begin{aligned} l^{(1)} &= (\overbrace{-1\,-1}^{l_0^{(1)}};\ 1\,1\ 0\,0 \ldots 0\,0) \\ l^{(2)} &= (-1\,-1;\ 0\,0\ 1\,1\ \ \ 0\,0) \\ l^{(l+1)} &= (-1,-1;\ \underbrace{0\,0\ 0\,0\ \ \ 1\,1}_{2l+2}) \end{aligned}$$

with solution Frobenius deformed solutions

$$X^0(\underline{t},\underline{\varepsilon}) = \sum_{n_1,\ldots n_{l+1}\geq 0} c(\underline{n}+\underline{\varepsilon})\, \underline{z}^{\underline{n}+\underline{\varepsilon}}$$

$n! \doteq \Gamma(n+1)$

$$c(\underline{n}) = \frac{\prod_{j=1}^{2} \left(-\sum_{k=1}^{l+1} l_{0j}^{(k)} n_k\right)!}{\prod_{i=1}^{2l+2} \left(\sum_{k=1}^{l+1} l_i^{(k)} n_k\right)!}$$

- $X^i_1(\underline{t}) = \frac{1}{2\pi i}\partial_{\varepsilon_i} X^0\big|_{\varepsilon=0} = \frac{X^0(t)}{2\pi i}\log(t_i) + \sigma_i \quad X^{\tau}_2 = \partial_{\varepsilon_i}\partial_{\varepsilon_j} X^0 \ldots$

yields the higher logarithmic solutions and the mirror map:

$$S^k = \frac{X^k}{X^0} = \frac{1}{2\pi i}\log(t_i) + \ldots \qquad (25)$$

- The $\hat{\Gamma}(TX) = 1 + \frac{c_2}{24} + i\frac{c_3\zeta(3)}{8\pi^3} + \frac{7c_2^2 - 4c_4}{5760} + \ldots$

$\Gamma(1-\frac{x}{2\pi i})$ class allows to fix the integral basis from homological mirror symmetry

$$\Pi_r^{(p)} = \int_{\hat{X}} e^{s^i J_i} \wedge \hat{\Gamma}(T\hat{X})\, ch(\mathcal{E}_r^{(p)\vee}) \qquad (26)$$

– The special solution can be found by the Ansatz

$n = \ell - 1$

$$X_\ell^i(\underline{t}) = \prod_{i=1}^{\ell+1} \log(t_i) \sum_{i=1}^{\ell+1} \frac{1}{\log(t_i)} + O(t_i) \qquad (27)$$

and in the general solution

$$I_{\text{Ban}} = \sum_{r=0}^{\ell_1} \sum_{s=1}^{d_r^{(\ell)}} X_r^s \; \lambda^{(\ell)}_{r,s} \qquad (28)$$

↑ The coefficient determined by a novel $\hat{\Gamma}$-Class

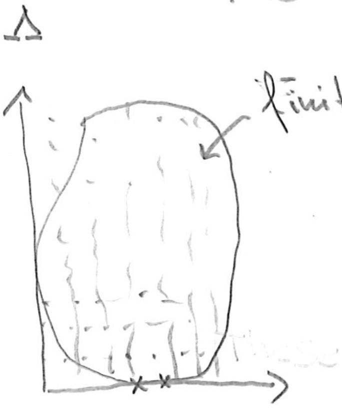
Further useful representations of I_G are the Baikov and the Schwinger representation

Thesis Christoph Nega Uni Bonn server

II.4 The Integration by parts method

J.M. Henn "Multiloop integrals in dimensional regularisation made "simple" Phys Rev Let [1304.1806]

The integration by parts identities



finite region

$$\int \prod_{r=1}^{\ell} \frac{d^D k_r}{i \pi^{D/2}} \frac{\partial}{\partial k^\mu_k} \left(q^\mu_e \prod_{j=1}^{\#P} \frac{1}{D_j^{\nu_j}} \right) = 0 \qquad (29)$$

↑ $\in \mathbb{Z}_{\geq 0}$ exponents in positive cone in lattice Δ

Propator power can be zero

$\int \prod_{r=1}^{\ell} \frac{d^D k_r}{i\pi^{D/2}} \prod_{j=1}^{\#P} \frac{1}{D_j^{\nu_j}}$

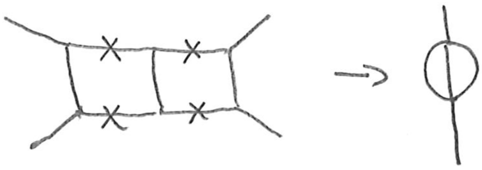
are differential relation that can be in terms of the Lorent invariant quantities t_i

Graph cohomology

$$\partial_{t_i} I_G = A_i(t_i, \varepsilon) I_G \qquad (30)$$

Gauss Manin connection for the Graph

Example: Double box



Subgraph

Banana 

tad pole
no momentum
integration