Relative Mori Cone $\overline{NE}(X/S)$
(KMM87)

Let $f: X \longrightarrow S$ be a projective morphism

Assume that X is normal

(Example: A family of proj varieties over S)

$\overline{NE}(X/S)$ is the convexe closed cone generated by complete curves contracted by $f: X \to S$

$\overline{NE}(X/S) \subseteq N_1(X/S)$
$= \{$ numerical classes of $\sum a_i C_i$, where $a_i \in \mathbb{R}$, C_i are curves contracted by f $\}$

Now can define relative ampleness, nefness, pseffness, bigness for divisors in X by numerical conditions.

Def: A $\mathbb{Q}$-Cartier divisor D in X is called f-relatively ample, or ample relatively over S if $C \cdot D > 0$ for any $C \in \overline{NE}(X/S) - \{0\}$.

D is nef if $C \cdot D \geq 0$ for $C \in \overline{NE}(X)$.

Prop: Let X be quasi-proj. D a Cartier divisor in X

Then D is f-relatively ample

(if has connected fibers and S normal)

$\Longleftrightarrow$ $\exists$ ample divisor H on X s.t.

$$D = H + f^*E, \quad \text{where } E$$

is a Cartier divisor on the base S

Def: D is f-relatively pseudo-eff (or big)

if $D = H + f^*E$ where H is pseudo-eff (or big) divisor in X

and E is a divisor in S.

Last time: Example: Let $f: X \to Y$ be a birational proj morphism. Assume that f contracts exactly one prime divisor $E \subseteq X$, and that E is $\mathbb{Q}$-Cartier.

Then $-E$ is f-relatively ample.

(Note: $E \geq 0$, thus $-E$ is kind of "negative" on X globally)

<u>Negativity Lemma</u>: (KM98. 3.39)
$f: X \to Y$ bir proj morphism. (X is $\mathbb{Q}$-factorial)
X, Y normal.
Assume that the exceptional locus of f is $\bigcup_{i=1}^{k} E_i$ where E_i are prime divisors

Let $D = \sum_{i=1}^{k} a_i E_i$ with $a_i \in \mathbb{Q}$

Then if D is f-ample, then $a_i < 0 \ \forall i$
if D is f-nef, then $a_i \leq 0 \ \forall i$.

Last time: We use this to prove the discrepancy increases during a divisorial contraction

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V 2 small contraction. (X, Δ) proj pair
R is an $(K_X + \Delta)$-negative extremal ray in $\overline{NE}(X)$

We have $f = \mathrm{cont}_R : X \to Z$ the contraction of R

Def: f is called small if the exceptional locus satisfies $\dim \mathrm{ex}(f) \leq \dim X - 2$

Remark ① In this case, $\mathrm{ex}(f)$ might have several irreducible components.

② Z is never $\mathbb{Q}$-factorial

Solution for ② is flip.

Let $f : X \to Z$ as before a small contraction.

the flip of f is $f^+ : X^+ \to Z$

with the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

$-(K_X+\Delta)$ is f-ample $\qquad (K_{X^+}+\Delta^+)$ f^+-ample

where φ is birational, $\Delta^+ = \varphi_* \Delta$

such that $\rho(X^+/Z) = \dim(N_1(X^+/Z))$

$$= \rho(X/Z) = 1$$

and that $(K_{X^+} + \Delta^+)$ is f^+-ample

Prop : Assume X $\mathbb{Q}$-factorial, (X,Δ) klt

Assume X^+ exists.

Then ① X^+ is $\mathbb{Q}$-factorial

② (X^+, Δ^+) is klt

indeed the discrepancy increases during a flip

③ $\rho(X) = \rho(X^+)$ Picard numbers

④ X^+ is unique, indeed

$$X^+ \simeq \mathrm{Proj}_Z \left(\bigoplus_{m \geq 0} f_* \mathcal{O}_X(mr(K_X + \Delta)) \right)$$

here $r > 0$ is an integer such that $r(K_X + \Delta)$ is integral Cartier.

In this case, $\bigoplus_{m \geq 0} f_* \mathcal{O}_X(mr(K_X + \Delta))$ is a finitely generated $\mathcal{O}_Z$-algebra.

Prop: the flip
$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] \\ & Z & \end{tikzcd}$$

exists if and only if

$\bigoplus_{m \geq 0} f_* \mathcal{O}_X(mr(K_X + \Delta))$ is finitely generated.

Fact: 3-dim flip exists by Mori (1982)

klt flip exists by ~~BCH~~ BCHM (~2010)

Log can flip exists Hacon-Xu, Birkar (~2015)

Exercise: Prove that the discrepancy increases during a flip (Hint: use common resolution

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "\varphi"] & & X^+ \end{tikzcd}$$

)

(KMM87, section "flip conjecture")

V3 Mori fibration. A contraction of extremal ray R is called a $(K_X+\Delta)$-Mori fibration if $\mathrm{ex}(f)=X$.

In this case, we have $f: X \longrightarrow Z$ such that Z is normal, f has connected fibers and $\dim Z < \dim X$.

Prop: In this case, if (X,Δ) $\mathbb{Q}$-factorial klt then Z is <u>$\mathbb{Q}$-factorial</u>, Z is klt.

then Z is $\underline{\mathbb{Q}\text{-factorial}}$, $\underline{Z \text{ is klt}}$.

ez to prove

Fujino ~2000
using Kawamata's
canonical bundle formula

$\underline{\text{V}4}$ MMP conjecture.

(X, Δ) proj klt $\mathbb{Q}$-factorial pair.

By cone thm, and existence of flips
We can performs a sequence of birational operation

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots$$

where each $X_i \dashrightarrow X_{i+1}$ is either a divisorial contraction or a flip.

Conjecture (finiteness of flips)
There is no infinite sequence of flips.

If this conjecture is true, then the sequence above end with $X_k = Y$

such that $X \underset{g}{\overset{bir}{\sim}} Y$, $\underline{\Delta} = g_* \Delta$, either $K_Y + \Delta$ is nef (minimal model)

or there is a $(K_Y + \Delta)$-Mori fibration

$Y \to Z$.

Note: Assume the conjecture, then $K_Y + \Delta$ is nef

$\Longleftrightarrow K_X + \Delta$ pseudo-eff

(BCHM): They prove flips "with scaling"

are always finite

(Kawamata, Shokurov, ~1990) 3-dim flips are always finite.

Exercise: $\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$ a flip, X, X^+ $\mathbb{Q}$-fac

Assume that $S \subseteq X$ is a prime divisor

such that S intersect $ex(f)$ but

$ex(f) \not\subseteq S$.

Then $S^+ \supseteq ex(f^+)$, where $S^+ = \varphi_* S$.

Hint: 1 Show that if C^+ a curve is contracted by f^+

Hint: ① Show that if C^+ is contracted by f^+ then $C^+ \cdot S^+ < 0$.

② Use $\rho(X/Z) = \rho(X^+/Z) = 1$ and relate S and $K_X + \Delta$

VI Singularities

VI.1 Surface Case.

Def (Minimal resolution): X normal surface.

There exists a "minimal resolution" of X $r: \tilde{X} \to X$ such that

if $W \to X$ is a resolution of sing then it factors through $\tilde{X}$, that is

$$W \to \tilde{X} \to X.$$

Remark: Only for surfaces.

Negativity Lemma in surface case.

$f: X \to Y$ be a birational proj morphism between normal surfaces

$ex(f) = \bigcup^k C_i$. Assume X $\mathbb{Q}$-fac

$i=1$

Then the matrix (intersection matrix)

$$\left[C_i \cdot C_j \right]_{1 \le i, j \le k} \text{ is def negative.}$$

Moreover, if $\sum_{i=1}^{k} a_i C_i$ is f-nef, then $a_i \le 0 \ \forall i$

Prop: $f: \tilde{X} \to X$ is the minimal resolution if and only if $K_{\tilde{X}}$ is f-nef.

(Kollár, relution of singularities)

Exercise, Assume K_X $\mathbb{Q}$-Cartier, let $f: \tilde{X} \to X$ minimal resolution. Write $K_{\tilde{X}} \equiv f^* K_X + \sum a_i C_i$. Show that $a_i \le 0$ for all i.

Prop (KM Sect 4): (X, Δ) surface pair, $\Delta \ge 0$.

(X, Δ) is terminal $\iff$ X is smooth, $\text{mult}_x \Delta < 1$ for all $x \in X$.

(X, Δ) is canonical $\iff$ either X is smooth, $\text{mult}_x \Delta \le 1$ for all $x \in X$.

or X sing, $\Delta = 0$
at singular point.
(or $\mathrm{mult}_x \Delta = 0$ if x singular)

Canonical surface singularities are also Du Val singularities (well classified)

1. can surf sing, K_X is always Cartier.
2. ———, X is always $\mathbb{Q}$-fac.
3. ——— are hypersurface sing
   which means locally, $x \in X$ is locally analytically iso to $(0 \in H \subseteq \mathbb{C}^3)$
   where H is a hypersurface, singular at 0.
4. ——— are quotient singularities
   that is $(x \in X)$ is locally analytically iso to $0 \in \mathbb{C}^2/G$, where $G \leq GL_2(\mathbb{C})$ finite subgroup, whose elements never fix hyperplane in $\mathbb{C}^2$, except $Id \in G$.

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klt surface sing (classification: [KM98 ...)

klt surface sing (classification in (KM98, Sect 4))

Prop: klt surface sing are all quotient sing.

Prop: Ever surface quotient sing is klt.

Thm: Surface case. klt $\Longleftrightarrow$ quotient sing.

Construction of birational model

Let $(x \in X)$ be a surface klt sing.

Let $r: \tilde{X} \longrightarrow X$ be minimal resolution.

Then we write $K_{\tilde{X}} = r^* K_X + \sum_{i=1}^{s} a_i C_i$

where the C_i are exceptional curves, and $a_i \leq 0$.

(Note: if one $a_i < 0$, then all $a_i < 0$)

Let $J \subseteq \{1, \dots, s\}$.

up to renumbering, we assume $J = \{1, \dots, k\}$.

Then there is a birational model Y

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[rr, bend right] & Y \arrow[r, "g"] & X \end{tikzcd}$$

r

such that f contracts exactly $C_1, \cdots, C_k$

Moreover, Y has klt sing

Proof : We can prove it by using classification

Here, we will prove by running MMP

(r-relative MMP)

Let $\tilde{D} = \sum_{i=1}^{s} (-a_i C_i)$, then $K_{\tilde{X}} + \tilde{D} \equiv r^* K_X$

Let $\tilde{D}' = \sum_{i=1}^{k} C_i$ (Note that $k \le s$)

For $1 \gg \varepsilon > 0$, $(\tilde{X}, \tilde{D} + \varepsilon \tilde{D}')$ is klt

Then we set $\tilde{\Delta} = \tilde{D} + \varepsilon \tilde{D}'$,

we run a r-relative MMP for $(\tilde{X}, \tilde{\Delta})$

Then we obtain

$$\begin{tikzcd}
(\tilde{X}, \tilde{\Delta}) \arrow[r, "f"] \arrow[dr, "r"'] & (Y, \Delta) \arrow[d, "g"] \\
& X
\end{tikzcd}$$

Exercise : Show that f contracts $C_1, \cdots, C_k$

(Hint : use negativity lemma)