2021-9-22 Kähler geometry

Correction

<u>Def</u> (M, g, J) with $d\omega = 0$ is called an almost Kähler manifold.

<u>Def</u> If further J is integrable, (M, g, J) is called a <u>Kähler manifold</u>, ω is called the <u>Kähler form</u>.

$[\omega] \in H^2_{DR}(M)$ is called the <u>Kähler class</u>.

<u>Def</u> Sometimes, a complex manifold admitting a Kähler metric is called a Kähler manifold.

<u>Def</u> M is non-Kähler $\overset{\text{def}}{\Longleftrightarrow}$ M does not admit a Kähler metric

Ex. If $H^2_{DR}(M, \mathbb{R}) = 0$ then M is non-Kähler. (non-symplectic)

$\boxed{\text{Extend } g, D, R \text{ and etc in the } \mathbb{C}\text{-linear way.}}$ (2)

$z^1, \dots, z^m$ local holo coordinates.

$\overline{z^1}, \dots, \overline{z^m} \longrightarrow z^{\bar{1}}, \dots, z^{\bar{m}}$ (new notations)

$\{A, B, C, \dots\} = \{1, \dots, m, \bar{1}, \dots, \bar{m}\}$

$$\frac{\partial}{\partial \overline{z^i}} = \frac{\partial}{\partial z^{\bar{i}}}, \quad d\overline{z^i} = dz^{\bar{i}}$$

$$g_{AB} = g\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right)$$

Lemma $\quad g_{ij} = 0, \quad g_{\bar{i}\bar{j}} = 0$

(:) $$g_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j}\right)$$

$$= g\left(\sqrt{-1}\frac{\partial}{\partial z^i}, \sqrt{-1}\frac{\partial}{\partial z^j}\right) = -g_{ij}$$

$$\therefore g_{ij} = 0$$

(:)

So, we only have $g_{i\bar{j}} = g_{\bar{j}i}$

Lemma $\quad \omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}$

$$\omega\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(J\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right)$$

$$= g\left(\sqrt{-1}\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = \sqrt{-1}\, g_{ij} = 0$$

$$\omega\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right) = 0.$$

$$\omega\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = g\left(J\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)$$

$$= \sqrt{-1}\, g_{i\bar{j}}$$

$$\therefore \omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

$$\nabla_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} = \Gamma_{AB}^{C} \frac{\partial}{\partial z^C}$$

Γ_{AB}^{C} is called the Christoffel symbol.

Torsion-free

$$\Longleftrightarrow \nabla_X Y - \nabla_Y X = [X, Y]$$

$$\nabla_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} - \nabla_{\frac{\partial}{\partial z^B}} \frac{\partial}{\partial z^A} = \left[\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right] = 0.$$

$$\therefore\ \Gamma^C_{AB} = \Gamma^C_{BA}$$ "torsion-free symmetric"

As we extended $\mathbb{C}$-linearly

$$\overline{\Gamma^C_{AB}} = \Gamma^{\bar C}_{\bar A \bar B}.$$

Prop Let (M, J) be a complex manifold and g a Hermitian metric. Then the following are equivalent:

(i) $d\omega = 0$, i.e. g is Kähler

(ii) $\nabla J = 0$

(iii) $\Gamma^C_{AB} = 0$ possibly except for Γ^k_{ij} and $\Gamma^{\bar k}_{\bar i \bar j}$.

Proof (ii) $\Rightarrow$ (i) $\nabla_X (JY) = (\nabla_X J)Y + J\nabla_X Y$

$$d\omega(X, Y, Z) = X\omega(Y, Z) + Y\omega(Z, X) + Z\omega(X, Y)$$
$$-\omega([X,Y], Z) - \omega([Y,Z], X) - \omega([Z,X], Y)$$
$$= X g(JY, Z) + Y g(JZ, X) + Z g(JX, Y)$$
$$- g(J[X,Y], Z) - g(J[Y,Z], X) - g(J[Z,X], Y)$$

$$= g((\nabla_X J)Y, Z) + g(J\nabla_X Y, Z) + g(JY, \nabla_X Z) \quad {}^{-g(Y, J\nabla_X Z)}$$ ⑤

$$+ g((\nabla_Y J)Z, X) + g(J\nabla_Y Z, X) + g(JZ, \nabla_Y X)$$

$$+ g((\nabla_Z J)X, Y) + g(J\nabla_Z X, Y) + \underline{g(JX, \nabla_Z Y)} \quad {}^{-g(X, J\nabla_Z Y)}$$

$$- g(J[X,Y], Z) - g(J[Y,Z], X) - g(J[Z,X], Y)$$

$$= g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y)$$

$$\therefore \nabla J = 0 \implies d\omega = 0$$

$$(\nabla_X J)(JY) = \nabla_X(J^2 Y) - J\nabla_X J \ldots$$

$(i) \implies (ii)$

$$0 = d\omega(X, Y, Z) - d\omega(X, JY, JZ)$$

$$= g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y)$$

$$- g((\nabla_X J)JY, JZ) - g((\nabla_{JY} J)JZ, X) - g((\nabla_{JZ} J)X, JY)$$

$$= g(\nabla_X(JY) - J\nabla_X Y, Z) + g(\nabla_Y(JZ) - J\nabla_Y Z, X)$$

$$+ g(\nabla_Z(JX) - J\nabla_Z X, Y) + g(\nabla_X Y - J\nabla_X(JY), JZ)$$

$$- g(\nabla_{JY}(-Z) - J\nabla_{JY}(JZ), X) - g(\nabla_{JZ}(JX) - J\nabla_{JZ}X, JY)$$

⑥

$$= 2g(\nabla_X Y, JZ) + 2g(\nabla_X(JY), Z)$$

$$+ g(\nabla_Z(JX), Y)_{\triangle} + g(\nabla_Z X, JY)_{\square}$$

$$+ g(J\nabla_Y(JZ), JX)_{④} + g(\nabla_Y Z, JX)_{①}$$

$$+ g(J\nabla_{JY} Z, JX)_{③} - g(\nabla_{JY} JZ, JX)_{②}$$

$$- JZ\, g(X, Y)_{⑤} + g(JX, \nabla_{JZ} JY)_{②}$$

$$+ JZ\, g(X, Y)_{⑤} - g(JX, J\nabla_{JZ} Y)_{④}$$

$$\cancel{2g(JX, Y)}$$
$$- g(JX, \nabla_Z Y)_{①}$$
$$+ 2g(\cancel{X}, JY)$$
$$- g(JX, J\nabla_Z JY)_{③}$$

$$= 2g((\nabla_X J)Y, Z)$$

$$+ g(JX, [Y, Z]_{①} - [JY, JZ]_{②} + J[JY, Z]_{③} + J[Y, JZ]_{④})$$

$$= 2g((\nabla_X J)Y, Z) + g(JX, \underbrace{N(Y, Z)}_{\overset{\shortparallel}{0}})$$

J integrable

$$= 2g((\nabla_X J)Y, Z)$$

$\therefore\ d\omega = 0 \Longrightarrow \nabla J = 0.$

(ii) $\nabla J = 0 \iff \nabla(JX) = J\nabla X$

$\iff \nabla \circ J = J \circ \nabla$

(iii) $\Gamma^C_{AB} = 0$ except for Γ^k_{ij}, $\Gamma^{\bar{k}}_{\bar{i}\bar{j}}$

(iii) $\implies$ $\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \Gamma^k_{ij} \frac{\partial}{\partial z^k} + \underbrace{\Gamma^{\bar{k}}_{ij}}_{0} \frac{\partial}{\partial \bar{z}^k} = \Gamma^k_{ij} \frac{\partial}{\partial z^k}$

$$\nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^j} = \underbrace{\Gamma^k_{\bar{i}j}}_{0} \frac{\partial}{\partial z^k} + \underbrace{\Gamma^{\bar{k}}_{\bar{i}j}}_{0} \frac{\partial}{\partial \bar{z}^k} = 0$$

$$\therefore \nabla_X \frac{\partial}{\partial z^i} \in C^\infty(T'M) \quad \text{for } {}^\forall X$$

$$J \nabla_X \frac{\partial}{\partial z^i} = \sqrt{-1}\, \nabla_X \frac{\partial}{\partial z^i}$$

On the other hand

$$\nabla_X\left(J \frac{\partial}{\partial z^i}\right) = \nabla_X\left(\sqrt{-1} \frac{\partial}{\partial z^i}\right) = \sqrt{-1}\, \nabla_X \frac{\partial}{\partial z^i}$$

$$\therefore \nabla_X\left(J \frac{\partial}{\partial z^i}\right) = J \nabla_X \frac{\partial}{\partial z^i}$$

In the same way

$$\nabla_X\left(J \frac{\partial}{\partial \bar{z}^i}\right) = J \nabla_X \frac{\partial}{\partial \bar{z}^i}$$

$\therefore\ \nabla \circ J = J \circ \nabla \qquad \therefore\ \nabla J = 0$ (ii)

(ii) $\Rightarrow$ (iii)

$$\nabla_{\frac{\partial}{\partial z^i}} \overbrace{J \frac{\partial}{\partial z^j}}^{\sqrt{-1}\frac{\partial}{\partial z^j}} = \sqrt{-1}\, \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \sqrt{-1}\left(\Gamma_{ij}^{k} \frac{\partial}{\partial z^k} + \Gamma_{ij}^{\bar{k}} \underbrace{\frac{\partial}{\partial \bar{z}^k}}\right)$$

$\| $ (ii)

$$J \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = J\left(\Gamma_{ij}^{k} \frac{\partial}{\partial z^k} + \Gamma_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}\right)$$

$$= \sqrt{-1}\, \Gamma_{ij}^{k} \frac{\partial}{\partial z^k} - \underbrace{\sqrt{-1}\, \Gamma_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}}$$

$$\Gamma_{ij}^{\bar{k}} = 0 \qquad \therefore\ \Gamma_{i\bar{j}}^{k} = \overline{\Gamma_{\bar{i}j}^{\bar{k}}} = 0$$

$$\nabla_{\frac{\partial}{\partial z^i}} \overbrace{J \frac{\partial}{\partial \bar{z}^j}}^{-\sqrt{-1}\frac{\partial}{\partial \bar{z}^j}} = -\sqrt{-1}\left(\Gamma_{i\bar{j}}^{k} \frac{\partial}{\partial z^k} + \Gamma_{i\bar{j}}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}\right)$$

$\|$ (ii)

$$J \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar{z}^j} = J\left(\Gamma_{i\bar{j}}^{k} \frac{\partial}{\partial z^k} + \Gamma_{i\bar{j}}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}\right)$$

$$= \sqrt{-1}\, \Gamma_{i\bar{j}}^{k} \frac{\partial}{\partial z^k} - \sqrt{-1}\, \Gamma_{i\bar{j}}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}$$

$$\therefore\ \Gamma_{i\bar{j}}^{k} = 0 = \Gamma_{\bar{j}i}^{k} \qquad \Gamma_{i\bar{j}}^{\bar{k}} = 0 = \Gamma_{\bar{j}i}^{\bar{k}}$$

$\ddot\smile$

$$g(JX, JY) = g(X, Y) \iff g \text{ Hermitian}$$

$$g_{ij} = g_{\bar{i}\bar{j}} = 0$$

since g is Riem metric, so sym.

Rem $(g_{j\bar{i}}) = (g_{\bar{i}j}) = \overline{(g_{i\bar{j}})}$, $(g_{i\bar{j}})$ Hermitian matrix

$$\frac{\partial}{\partial z^i} g_{j\bar{k}} = g\left(\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}\right) = \Gamma^l_{ij}\, g_{l\bar{k}}$$

$g_{\bar{k}l}$

$$g^{i\bar{k}} g_{\bar{k}j} = \delta^i{}_j, \quad \begin{cases} g = (g_{\bar{i}j}) \\ g^{-1} = (g^{i\bar{j}}) \end{cases}$$

$$\Gamma^k_{ij} = g^{k\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^i}, \qquad \Gamma^i_{kj} = g^{i\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^k}$$

$$\left(\Gamma^i_{kj}\, dz^k\right) = g^{-1} \partial g = \left(g^{i\bar{l}} \frac{\partial g_{\bar{l}j}}{\partial z^k} dz^k\right)$$

$$\theta^i{}_j = \Gamma_k{}^i{}_j \, dz^k$$

$\theta = (\theta^i{}_j)$ connection 1-form
connection matrix.

$= g^{-1} \partial g$ ⟵ Remember

$\Theta \overset{\text{def}}{=} d\theta + \theta \wedge \theta$ Ω ω

curvature form
曲率

$d = \partial + \bar{\partial}$

$\theta = g^{-1} \partial g$

$\partial (g^{-1})$

$= -g^{-1} \partial g \cdot g^{-1}$

exchange.

$$= (\partial + \bar{\partial}) \theta + \theta \wedge \theta$$

$$= \bar{\partial} \theta + \partial \theta + \theta \wedge \theta$$

$$= \bar{\partial} \theta - \cancel{g^{-1} \partial g \cdot g^{-1} \wedge \partial g}$$

$$+ \cancel{g^{-1} \partial g \wedge g^{-1} \partial g}$$

$$= \bar{\partial} \theta$$ ⟵ Remember

Prop $\underline{e = \left(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^m}\right)}$

$$e\,\Theta(X,Y) = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) e$$

$$\Theta(X,Y) = (d\theta + \underline{\theta\wedge\theta})(X,Y)$$

$$= X\theta(Y) - Y\theta(X) - \theta([X,Y])$$

$$+ \theta(X)\theta(Y) - \theta(Y)\theta(X)$$

On the other hand $\quad e \to e$

$$\text{RHS} = \nabla_X \nabla_Y e - \nabla_Y \nabla_X e - \nabla_{[X,Y]} e$$

$$= \nabla_X(e\,\theta(Y)) - \nabla_Y(e\,\theta(X)) - e\cdot\theta([X,Y])$$

$$= e\,\theta(X)\cdot\theta(Y) + e\,X\theta(Y)$$

$$- e\,\theta(Y)\theta(X) - e\,Y\theta(X) - e\,\theta([X,Y])$$

$$= e\,\Theta(X,Y)$$

$$\therefore \boxed{R(X,Y)e = e\,\Theta(X,Y)}$$

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If we use 国算清

$$\mathbb{X} = (x^1 \cdots x^n) \begin{pmatrix} e^1 \\ \vdots \\ e^n \end{pmatrix} \qquad \nabla e = \theta e$$

e

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) e$$

$$= \nabla_X (\theta(Y) e) - \nabla_Y (\theta(X) e) - \theta([X,Y]) e$$

$$= X\theta(Y) e + \theta(Y)\theta(X) e$$

$$- Y\theta(X) - \theta(X)\theta(Y) e - \theta([X,Y]) e$$

$$= (d\theta(X,Y)) e - (\theta \wedge \theta)(X,Y) e.$$

$$\therefore \Theta = d\theta - \theta \wedge \theta$$

Shiing-Shen Chern 陳省身

Complex manifolds without potential theory

P principal G-bundle $\qquad \theta = (\theta^i_j)$

with left G-action ...

Western culture $\qquad \mathbb{X} = (e_1 \cdots e_n) \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$

P has right G-action $\qquad \Theta = d\theta + \theta \wedge \theta$

$\Theta = \Theta^{\alpha}{}_{\beta i\bar{j}}\, dz^i \wedge d\bar{z}^j$ vector bundles

$\Theta = R^k{}_{l i\bar{j}}\, dz^i \wedge d\bar{z}^j$ $T'M$

Kähler

Riemann

$R_{ij k\bar{l}} = 0$

$\because \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar{z}^l} = \Gamma^A_{j\bar{l}} \frac{\partial}{\partial z^A} = 0$

$\nabla_{[\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}]} = 0$

$R_{ijk\bar{l}} = g\left(\frac{\partial}{\partial z^k}, \left(\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial z^j}} - \nabla_{\frac{\partial}{\partial z^j}} \nabla_{\frac{\partial}{\partial z^i}}\right) \frac{\partial}{\partial \bar{z}^l}\right)$

$= 0 \quad \therefore$

Similarly $R_{ij\bar{k}\bar{l}} = 0$

$\therefore R_{ij\bar{l}k} = -R_{ijk\bar{l}} = 0$

$\therefore R_{\bar{i}\bar{j}\bar{k}l} = \overline{R_{ijk\bar{l}}} = 0$

Also $R_{ijkl} = g\left(\frac{\partial}{\partial z^k}, \left(\nabla_{\frac{\partial}{\partial z^i}} \nabla_{\frac{\partial}{\partial z^j}} - \nabla_{\frac{\partial}{\partial z^j}} \nabla_{\frac{\partial}{\partial z^i}}\right)\frac{\partial}{\partial z^l}\right) = 0$

type (1,0)

$g_{kl} = 0$.

$R_{i\bar{j}k\bar{l}} = 0$.

$\boxed{\text{Thm}}$

We only have possible non-zero components as $R_{i\bar{j}k\bar{l}}$ or its symmetries.