PART II. Deligne–Lusztig theory for finite reductive groups

Chapter 8. Deligne–Lusztig theory

- G connected reductive algebraic group
- $F : G \longrightarrow G$ Frobenius endomorphism $/\mathbb{F}_q$
- $G^F = \{ g \in G \mid F(g) = g \}$ (finite reductive group)
- B_0 : F-stable Borel subgroup
  $\cup$
  T_0 : F-stable maximal torus; $B_0 = T_0 \ltimes U_0$

- $W = N_G(T_0)/T_0$: Weyl group
- $S = \{ s \in W \mid \dim B_0 s B_0 - \dim B_0 = 1 \}$
- $W = \langle S \rangle$ and $s^2 = 1$ for all $s \in S$.

- $\nabla(G,F) = \{ (T,\theta) \mid F(T) = T \text{ and } \theta : T \to K^\times \}$
- $(T,\theta) \equiv (T',\theta') \iff \exists d \geq 1$ s.t. $(T, \theta \circ N_{F^d/F})$ and $(T', \theta' \circ N_{F^d/F})$ are G^{F^d}-conjugate

- $\mathcal{E}(G^F, \mathcal{X}) = \{ \chi \in \operatorname{Irr} G^F \mid \exists (T,\theta) \in \mathcal{X}$ s.t. $\langle R_T^G(\theta), \chi \rangle \neq 0 \}$

(Lusztig series)

Theorem 8.19 + 8.20 (Deligne-Lusztig 1976)

- $\operatorname{Irr} G^F = \dot{\bigcup}_{\mathcal{X} \in \nabla(G,F)/\equiv} \mathcal{E}(G^F, \mathcal{X})$

- $\mathcal{E}(G^F, \mathcal{X}) = \{ E \in \operatorname{Irr} G^F \mid \exists (T,\theta) \in \mathcal{X},$
$\exists B \supset T$ Borel s.t. $\operatorname{Hom}_{G^F}(E, \mathcal{R}^G_{T \subset B} K_\theta) \neq 0 \}$

Remark 8.21. Since all maximal tori are conjugate in G, the set TOR of maximal tori of G is in bijection with $G/N_G(T_0)$. So it can be endowed with a structure of algebraic variety which is:

- defined over $\mathbb{F}_q$
- acted on transitively by G.

Therefore (proposition 8.3(b))

$$(8.22) \qquad TOR^F/G^F \xleftrightarrow{\sim} H^1(F, W)$$

because $N_G(T_0)^\circ = T_0$.

Moreover (Steinberg, 1968)

$$(8.23) \qquad |TOR^F| = |G^F|_p^2 \qquad \blacksquare$$

Theorem 8.20 is a consequence of Theorem 8.19 and the following result:

Proposition 8.24.

(a) $${}^*R_T^G(\chi_{KG^F}) = \varepsilon_T \varepsilon_G \frac{|G^F|_{p'}}{|T^F|} \chi_{KT^F}$$

(b) $$\chi_{KG^F} = \frac{1}{|G^F|_p} \sum_{T \in TOR^F} \varepsilon_G \varepsilon_T R_T^G(\chi_{KT^F})$$

Proof. (a) Let B be a Borel subgroup containing T.

$$\begin{aligned}
{}^*R_T^G(\chi_{KG^F})(t) &= \frac{1}{|G^F|} \sum_{g \in G^F} Tr^*_{Y_B}(g,t)\, \chi_{KG^F}(g^{-1}) \\
&= Tr^*_{Y_B}(1,t) \\
&= Tr^*_{Y_B^t}(1) \qquad \text{(see 3.10(b))} \\
&= \begin{cases} 0 & \text{if } t \neq 1 \\ \varepsilon_G \varepsilon_T |G^F|_{p'} & \text{if } t = 1 \qquad \text{(see 8.15)} \end{cases} \\
&= \varepsilon_G \varepsilon_T \frac{|G^F|_{p'}}{|T^F|} \chi_{KT^F}(t).
\end{aligned}$$

Proposition 8.24.

(a) $${}^*R_T^G\left(\chi_{KG^F}\right) = \varepsilon_T\,\varepsilon_G\,\frac{|G^F|_{p'}}{|T^F|}\,\chi_{KT^F}$$

(b) $$\chi_{KG^F} = \frac{1}{|G^F|_p}\sum_{T\in \mathrm{Tor}^F}\varepsilon_G\,\varepsilon_T\,R_T^G(\chi_{KT^F})$$

Proof of (b). Let f be the right-hand side.

- $\langle \chi_{KG^F}, \chi_{KG^F}\rangle_{G^F} = |G^F|$
- $$\langle \chi_{KG^F}, f\rangle_{G^F} = \frac{1}{|G^F|_p}\sum_{T\in\mathrm{Tor}^F}\varepsilon_G\varepsilon_T\langle \chi_{KG^F}, R_T^G\chi_{KT^F}\rangle_{G^F}$$
$$= \frac{1}{|G^F|_p}\sum_{T\in\mathrm{Tor}^F}\varepsilon_G\,\varepsilon_T\,\langle {}^*R_T^G\chi_{KG^F}, \chi_{KT^F}\rangle_{T^F}$$
(adjunction 8.5)
$$= \frac{1}{|G^F|_p}\sum_{T\in\mathrm{Tor}^F}|G^F|_{p'} \quad \text{(see (a))}$$
$$= \frac{|G^F|_p^2\cdot|G^F|_{p'}}{|G^F|_p} = |G^F| \quad \text{(see (8.23))}$$

- $$\langle f, f\rangle = \frac{1}{|G^F|_p^2}\sum_{T,T'\in\mathrm{Tor}^F}\varepsilon_T\,\varepsilon_{T'}\langle R_T^G\chi_{KT^F}, R_{T'}^G\chi_{KT'^F}\rangle$$
$$= \frac{1}{|G^F|_p^2}\sum_{\substack{T,T'\in\mathrm{Tor}^F\\ T\underset{G^F}{\sim}T'}}\left|N_{G^F}(T)/T^F\right|\,\langle\chi_{KT^F},\chi_{KT^F}\rangle_{T^F}$$
(by the Mackey formula)
$$= \frac{1}{|G^F|_p^2}\sum_{T\in\mathrm{Tor}^F}|N_{G^F}(T)|\cdot\underbrace{\#\{G^F\text{-orbit of }T\}}_{\frac{|G^F|}{|N_{G^F}(T)|}}$$
$$= \frac{1}{|G^F|_p^2}\sum_{T\in\mathrm{Tor}^F}|G^F|$$
$$= |G^F| \quad \text{(see (8.23))}.$$

So $\langle f-\chi_{KG^F}, f-\chi_{KG^F}\rangle = 0$. ∎

Question. Is there a geometric proof of (b)?

Definition 8.25. Let

$$\mathcal{X}_1 = \{ (T, 1_{T^F}) \mid T \in \mathrm{TOR}^F \} \in \nabla(G,F)/_{\equiv}$$

The elements of $\mathrm{UNIP}(G^F) := \mathcal{E}(G^F, \mathcal{X}_1)$ are called the unipotent characters.

Example 8.26. The trivial character is unipotent: indeed, by example 8.9,

$$\langle 1_{G^F}, R_T^G 1_{T^F} \rangle_{G^F} = \langle {}^*R_T^G 1_{G^F}, 1_{T^F} \rangle_{T^F}$$
$$= 1. \blacksquare$$

Example 8.27. $\mathrm{UNIP}(SL_2(\mathbb{F}_q)) = \{ 1_{G^F}, St \}. \blacksquare$

Theorem 8.28 (Lusztig, 1984)

(a) There exists a finite set $\mathrm{Unip}(W,F)$ and a map $\mathrm{DEG} : \mathrm{Unip}(W,F) \longrightarrow \mathbb{Q}[t]$ which are both independent on q and such that there is a natural bijection

$$\gamma : \mathrm{Unip}(W,F) \xrightarrow{\sim} \mathrm{UNIP}(G^F)$$

satisfying

$$\gamma_i(1) = (\mathrm{DEG}\ i)(q).$$

(b) If $\mathcal{X} \in \nabla(G,F)/_{\equiv}$, there exists a reductive group $G_{\mathcal{X}}$ defined over $\mathbb{F}_q$ such that there is a "natural" bijection

$$\mathrm{Jor}_{\mathcal{X}} : \mathrm{UNIP}(G_{\mathcal{X}}^F) \xrightarrow{\sim} \mathcal{E}(G^F, \mathcal{X})$$

such that

$$\mathrm{Jor}_{\mathcal{X}}(\gamma)(1) = \frac{|G^F|_{p'}}{|G_{\mathcal{X}}^F|_{p'}} \cdot \gamma(1).$$

(Jordan decomposition).

Chapter 9. Application to GL_n

In this chapter

$$G = GL_n(\mathbb{F})$$

and $F(a_{ij}) = (a_{ij}^q)$

so that $G^F = GL_n(\mathbb{F}_q)$.

9.A. Conjugacy classes

If $g \in GL_n(\mathbb{F}_q)$, we denote by M_g the $\mathbb{F}_q[t]$-module whose underlying space is $(\mathbb{F}_q)^n$ and on which t acts through g.

Then $g \sim_{G^F} g' \iff M_g \simeq M_{g'}$. So

$$\{\text{conj. classes in } GL_n(\mathbb{F}_q)\}$$
$$\updownarrow 1$$
$$\left\{\begin{array}{c}\text{iso. classes of } \mathbb{F}_q[t]\text{-modules of dim. } n \\ \text{on which } t \text{ is invertible}\end{array}\right\} = \mathcal{M}od_n(q)$$

Let $\mathrm{IrrPol}^{\#}(q)$ denote the set of monic irr. pol. $\neq t$ with coefficients in $\mathbb{F}_q$ and let $\mathcal{S}_n(q)$ denote the set of maps

$$\lambda : \mathrm{IrrPol}^{\#}(q) \longrightarrow \{\text{partitions}\}$$

such that $\sum_{P \in \mathrm{IrrPol}^{\#}(q)} |\lambda_P| \cdot \deg(P) = n$

Here, if we write $\lambda_P = (\lambda_{P,1} \geqslant \lambda_{P,2} \geqslant \dots)$, then

$$|\lambda_P| = \sum_{i \geqslant 1} \lambda_{P,i}.$$

(9.1)
$$\begin{array}{rcl} \mathcal{S}_n(q) & \xrightarrow{\sim} & \mathcal{M}od_n(q) \\ \lambda & \longmapsto & \bigoplus_{P \in \mathrm{IrrPol}^{\#}(q)} \left(\bigoplus_{i \geqslant 1} \mathbb{F}_q[t] / \langle P^{\lambda_{P,i}} \rangle \right) \end{array}$$

(9.2) $|\{\text{conj. classes in } GL_n(\mathbb{F}_q)\}| = |\mathcal{S}_n(q)|$.

For $\lambda \in \mathcal{S}_n(q)$, we set

$$\begin{array}{rcl} |\lambda| : \mathrm{IrrPol}^{\#}(q) & \longrightarrow & \mathbb{Z}_{\geqslant 0} \\ P & \longmapsto & |\lambda_P| \end{array}$$

9.B. Some cuspidal representations

Let $c_n = (1, 2, \ldots, n) \in S_n = W \hookrightarrow GL_n(\mathbb{F}_q)$

$T_0 \simeq (\mathbb{F}^\times)^n$ and $T_0^{c_n F} \simeq \mathbb{F}_{q^n}^\times \simeq T_{c_n}^F$

(example (4) in §6.A)

Then $N_{G^F}(T_{c_n}^F) \simeq N_{G^{c_n F}}(T_0^{c_n F}) / T_0^{c_n F}$

$= C_{S_n}(c_n) = \langle c_n \rangle$

and c_n^{-1} acts on $T_0^{c_n F}$ by raising to the q-th power. Through this identification with $\mathbb{F}_{q^n}^\times$, c_n^{-1} is just the Frobenius.

$\theta \in \mathrm{Irr}(T_{c_n}^F) = \mathrm{Hom}_{gp}(T_{c_n}^F, K^\times)$ is

regular $\iff |\{\theta, \theta^q, \ldots, \theta^{q^{n-1}}\}| = n$

So there is a bijection between the set of $N_{G^F}(T_{c_n})$-orbits of regular characters of $T_{c_n}^F$ and the ~~set~~ set of orbits of length n in $\mathrm{Hom}_{gp}(\mathbb{F}_{q^n}^\times, K^\times)$ for the operation of raising to the q-th power.

But $\mathrm{Hom}_{gp}(\mathbb{F}_{q^n}^\times, K^\times) \xrightarrow{\sim} \mathbb{F}_{q^n}^\times$ ☹

so:

$$\begin{array}{ccc}
\theta_P & \in & \{N_{G^F}(T_{c_n})\text{-orbits of reg. char. of } T_{c_n}^F\} \\
\uparrow & & \updownarrow 1:1 \\
 & & \{F\text{-orbits of length } n \text{ in } \mathbb{F}_{q^n}^\times\} \\
 & & \updownarrow 1:1 \\
P & \in & \{\text{monic irr. pol. of deg. } n \text{ in } \mathbb{F}_q[t] \neq t\}
\end{array}$$

Note also that $\varepsilon_{T_{c_n}} = -1$ (check!) so

(9.2) $\gamma_P = (-1)^{n-1} R_{T_{c_n}}^G(\theta_P) \in \mathrm{Irr}_{cus}(G^F)$

Proof. Just need to check that T_{c_n} is not contained in an F-stable Levi comp. of an F-stable par. sub. $P \subsetneq G$. But these Levi complements are conjugate to

$GL_{n_1}(\mathbb{F}_q) \times \cdots \times GL_{n_r}(\mathbb{F}_q) \hookrightarrow GL_n(\mathbb{F}_q)$

with $n_1 + \cdots + n_r = n$ and $r \geqslant 2$; $n_i \geqslant 1$.

So this amounts to say that no conjugate of c_n is contained in $S_{n_1} \times \cdots \times S_{n_r} \subset S_n$. ∎

9.C. Harish-Chandra theory.

Let $\lambda \in \mathcal{P}_n(q)$:

$$\begin{tikzcd} \lambda : \mathrm{InPol}^{\#}(q) \arrow[r] \arrow[dr, "|\lambda|"'] & \{\text{partitions}\} \arrow[d] \\ & \mathbb{Z}_{\geq 0} \end{tikzcd}$$

We set
$$L_{|\lambda|} = \prod_{P \in \mathrm{InPol}^{\#}(q)} \left(GL_{\deg P}(\mathbb{F}) \right)^{|\lambda_P|} \subset GL_n(\mathbb{F})$$

$$\gamma_{|\lambda|} = \bigotimes_{P \in \mathrm{InPol}^{\#}(q)} \left(\gamma_P \otimes \cdots \otimes \gamma_P \right) \quad (|\lambda_P| \text{ times})$$
$$\in \mathrm{Irr}_{cus}\left(L^F_{|\lambda|}\right) \quad \text{by } (9.2).$$

Then
$$N_{GL_n(\mathbb{F}_q)}\left(L_{|\lambda|}, \gamma_{|\lambda|}\right) \Big/ L^F_{|\lambda|} \simeq \prod_P S_{|\lambda_P|}$$

$$\mathrm{Irr}\Big(\prod_P S_{|\lambda_P|} \Big) \xrightarrow{\ \sim\ } HC\left(GL_n(\mathbb{F}_q), L_{|\lambda|}, \gamma_{|\lambda|} \right) \quad \text{(see coro. 7.12)}$$
$$\chi_\lambda = \bigotimes_P \chi_{\lambda_P} \longmapsto \rho_\lambda$$

Theorem 9.3 (Green 1955; Deligne-Lusztig + Lusztig-Srinivasan 1976)

The injective map
$$\mathcal{P}_n(q) \longrightarrow \mathrm{Irr}\, GL_n(\mathbb{F}_q) \quad \text{is bijective.}$$
$$\lambda \longmapsto \rho_\lambda$$

Corollary 9.4.
$$\left| \mathrm{Irr}_{cus}\left(GL_n(\mathbb{F}_q)\right) \right| = \#\{\text{irr. pol. of degree } n\}$$
$$= \frac{1}{n} \Big(\sum_{d \mid n} \mu\left(\tfrac{n}{d}\right) q^d \Big) \quad ; \ \mu : \text{Möbius function.}$$