4-26/28/5-7

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Hirzebruch's signature theorem

$\Rightarrow \sigma(Q) = \langle L_k(p_1(Q), \cdots, p_k(Q)), [Q] \rangle$

$\overset{\text{Fact 2}}{=} s_k \langle p_k(Q), [Q] \rangle$

where $s_k = \frac{2^{2k}(2^{2k-1}-1)}{(2k)!} B_k$ is the coefficient of p_k

in the L_k-polonomial.

$\overset{\text{Fact 1}}{\Longrightarrow} \sigma(W) = \sigma(Q) = s_k \langle p_k(Q), [Q] \rangle$

$= \frac{n_k}{d_k} \langle p_k(Q), [Q] \rangle$ → a fraction in lowest terms

$\in \mathbb{Z}$

$\sigma(Q) \in \mathbb{Z} \Rightarrow d_k \mid \langle p_k(Q), [Q] \rangle$

So to show Prop 3.23, it suffices to find prime $p \neq 2$ s.t.

$p \mid t_k$ and $p \mid n_k$.

This is in fact a direct corollary of Facts 3.4.

Fact 3: [A00, Prop 1.6]
For all $k > 1$, there exists a prime $p > 2k+1$
s.t. $p \mid n_k$

Fact 4: If prime $p > 2k+1$ and $p \mid n_k$. ⟶ easy exercise!

Fact 4: If prime $p > 2k+1$ and $p \mid n_k$ $\longrightarrow$ easy exercise!

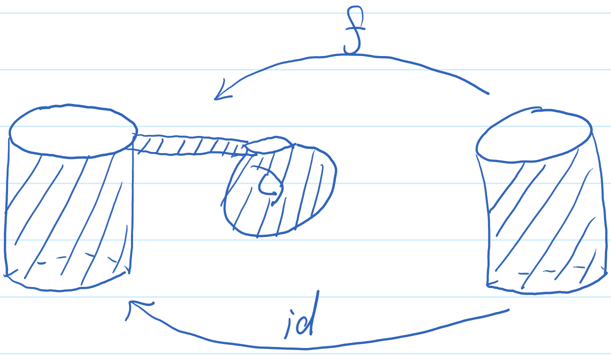
$\Longrightarrow p \mid t_k$.

$\square$

Pf Fact 2:

Q can be identified with

$(M \times I \natural W) \cup_{id \cup f} M \times I$ where $(M \times I \natural W) \cap M \times I = M \sqcup M$.



Apply M-V seq: for $i \geq 1$

$$H^{4i-1}(M \sqcup M; \mathbb{Q}) \xrightarrow{\delta} H^{4i}(Q;\mathbb{Q}) \xrightarrow{(j_1^*, j_2^*)} H^{4i}(M \times I; \mathbb{Q}) \oplus H^{4i}(M \times I \natural W; \mathbb{Q}) \quad (*)$$

$$p_i(Q) \longmapsto (j_1^* p_i(Q), \underbrace{j_2^* p_i(Q)}_{\overset{\text{claim}}{=} 0}).$$

where $j_1: M \times I \longrightarrow Q$, $j_2: M \times I \natural W \longrightarrow Q$ are inclusions.

Claim: $j_1^* p_i(Q) = 0$; $j_2^* p_i(Q) = 0$.

Pf claim:

$$j_1^* TQ = T(M \times I) = pr_1^* TM \oplus pr_2^* TI$$
$$= pr_1^* TM \oplus \varepsilon^1$$

$$\Rightarrow j_1^* p_i(Q) = pr_1^* p_i(M) = 0$$

since $M \times I \natural W \simeq M \times I \vee W$

$$H^{4i}(Q;\mathbb{Q}) \longrightarrow H^{4i}(M \times I \natural W; \mathbb{Q}) \xrightarrow[\cong]{(j_3^*, j_4^*)} H^{4i}(M \times I; \mathbb{Q}) \oplus H^{4i}(W;\mathbb{Q})$$

$$p_i(Q) \longmapsto j_2^* p_i(Q) \longmapsto (p_i(M \times I), p_i(W))$$

$\qquad\qquad\qquad\qquad \|\qquad\qquad\qquad\qquad \|$

$$p_i(Q) \longmapsto j_2^* p_i(Q) \longmapsto (p_i(M\times I),\ p_i(W))$$

$$j_2^* p_i(Q) = p_i(M\times I \sqcup W), \qquad p_i(M\times I) = 0, \qquad p_i(W) = 0 \to W \text{ is parallel.}$$

where j_3, j_4 are inclusions.

$\square$.

Claim + exactness of $(*)$ $\Rightarrow$ $p_i(Q) \in \operatorname{im}\delta$ for $i>0$.

Fact: $\forall\, x, y \in \operatorname{im}\delta \subset H^*(Q)$. $\Rightarrow$ $x \cup y = 0$ → see Rk below for pf.

$\Downarrow$

For $i_1, i_2 > 0$, $p_{i_1}(Q)\, p_{i_2}(Q) = 0$ $\square$

Rk: Consider the MV seq for

$$X = X_1 \cup X_2 \qquad \begin{array}{ccc} & X_1 & \\ \overset{j_1}{\swarrow} & & \overset{i_1}{\nwarrow} \\ & & X_1\cap X_2 \\ \underset{j_2}{\nwarrow} & & \underset{i_2}{\swarrow} \\ & X_2 & \end{array}$$

$$H^{*-1}(X_1\cap X_2) \xrightarrow{\delta} H^*(X) \xrightarrow{(j_1^*,\, j_2^*)} H^*(X_1)\oplus H^*(X_2) \xrightarrow{i_1^* - i_2^*} H^*(X_1\cap X_2)$$

$\forall\, v \in H^*(X)$, $u \in H^*(X_1\cap X_2)$ where $i = j_1\circ i_1 = j_2 \circ i_2$

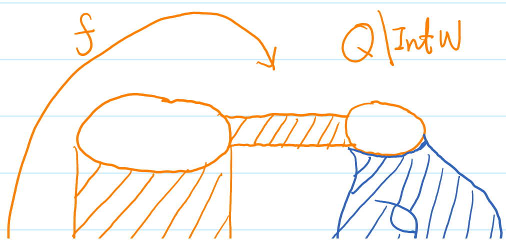
one has $\boxed{\delta(u \cup i^* v) = \delta u \cup v}$ — cf. [S, p.252, 12]

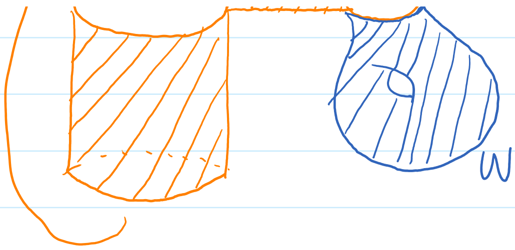
let $x = \delta u$, $v = y \in \operatorname{im}\delta$ $\Longrightarrow$ Fact

$i^*\delta = i_1^* j_1^* \delta = 0$

by exactness of MV seq.

pf Fact 1 $(\sigma(Q) = \sigma(W))$





Since $(Q \setminus \mathrm{Int}\, W) \cap W = \Sigma$

$$\Rightarrow Q = (Q \setminus \mathrm{Int}\, W) \cup_\Sigma W \approx \widehat{(Q \setminus \mathrm{Int}\, W)} \# \hat{W}$$

connected sum of top. mfds

where $\hat{W} \triangleq W \cup_\Sigma \mathrm{Cone}(\Sigma)$

$\widehat{(Q \setminus \mathrm{Int}\, W)} \triangleq (Q \setminus \mathrm{Int}\, W) \cup_\Sigma \mathrm{Cone}(\Sigma)$

top. mfds since $\mathrm{Cone}(\Sigma) \approx D^{4k}$

$$\Rightarrow \sigma(Q) = \sigma(\widehat{Q \setminus \mathrm{Int}\, W}) + \sigma(\hat{W})$$
$$= \sigma(\widehat{Q \setminus \mathrm{Int}\, W}) + \sigma(W)$$

Hence Fact 1

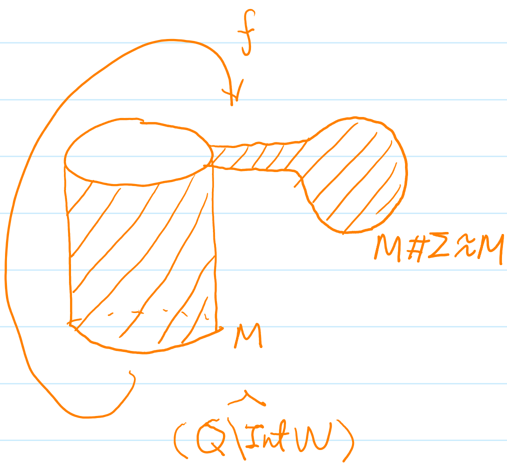
$\Uparrow$

$\sigma(\widehat{Q \setminus \mathrm{Int}\, W}) = 0$

$\Uparrow$

$\widehat{(Q \setminus \mathrm{Int}\, W)} \approx (M \times [0,1])_h \triangleq M_h$

with $h: M \circlearrowleft \approx$ orientation-preserving.



Claim $\sigma(M_h) = 0$

for any top. closed oriented mfd M and

$h: M \xrightarrow{\approx} M$ orientation-preserving

Pf Claim: Recall

Fact: Let V be a $\mathbb{R}$-vector space of even rank. and let

$$B: V\times V \longrightarrow \mathbb{R}$$

be a nondegenerate symmetric bilinear form.
If $\exists$ subspace $W\subset V$ with

$$\dim W = \tfrac{1}{2}\dim V$$

and $B(w_1, w_2)=0$ for all $w_1, w_2\in W$.
then $\sigma(B)=0$

cf. [MH, I. §6] Milnor, Husemoller, "Symmetric bilinear form".

So to show Claim, it suffices to find subspace

$$W\subset \boxed{H^{2k}((M\times I)_h)}$$

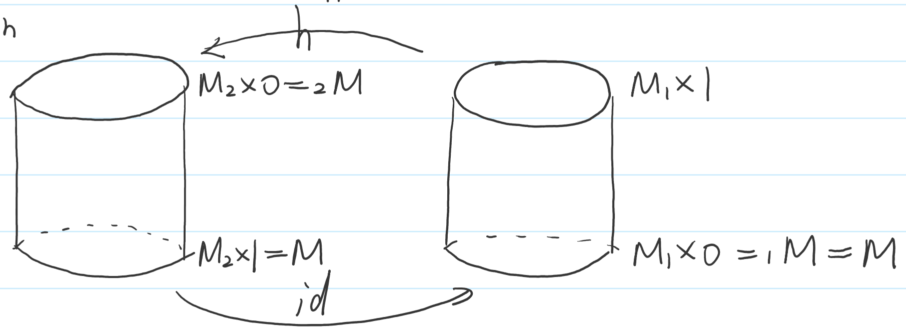
$\longrightarrow$! $H^*(\ )$ denotes cohomology with coeff. $\mathbb{R}$ in this proof.

s.t.

$$W\times W \xrightarrow{0} H^{4k}((M\times I)_h)$$
$$\forall\ (w_1,w_2) \longmapsto 0$$

$(M\times I)_h \overset{\text{identify}}{=} (M_1\times I)\cup(M_2\times I)$ as below:

$\cong$ M_h ; $\parallel$



By M-V seq $\rightsquigarrow$ long exact seq:

$$\xrightarrow{l^{j-1}} H^{j-1}({}_1M\sqcup{}_2M)\xrightarrow{\delta^{j-1}} H^j(M_h)\longrightarrow H^j(M_1\times I)\oplus H^j(M_2\times I)\underset{l^j}{\overset{i_1^*-i_2^*}{\underset{\cong}{\longrightarrow}}} H^j({}_1M\sqcup{}_2M)\longrightarrow$$

$$\Rightarrow\ 0\to \operatorname{im}\delta^{j-1}\to H^j(M_h)\to \ker l^j\to 0 \quad \text{exact.}$$

$\Rightarrow$ It suffices to show

$$\dim\operatorname{im}\delta^{2k-1} = \dim\ker l^{2k}$$

$\parallel$ $\dim \operatorname{im}\delta^{j-1} = \dim H^{j-1}({}_1M\sqcup{}_2M) - \dim\ker\delta^{j-1}$

$$\dim \operatorname{im} \delta^{2k-1} = \dim \ker \ell^{2k}$$

$$\Bigg\Downarrow$$

$$\begin{aligned} \dim \operatorname{im} \delta^{j-1} &= \dim H^{j-1}({}_1M \sqcup {}_2M) - \dim \ker \delta^{j-1} \\ &= \dim H^{j-1}({}_1M \sqcup {}_2M) - \dim \operatorname{im} \ell^{j-1} \\ &= \dim \operatorname{coker} \ell^{j-1} \\ &= \dim \ker \ell^{j-1} \end{aligned}$$

It suffices to show $\dim \ker \ell^{2k-1} = \dim \ker \ell^{2k}$. $(*)$.

To show $(*)$, compute ℓ^j as follows:

$$\begin{array}{ccc} H^j(M_1 \times I) \oplus H^j(M_2 \times I) & \xrightarrow[i_1^* - i_2^*]{\ell^j} & H^j({}_1M \sqcup {}_2M) \\ \| & & \| \\ H^j(M) \oplus H^j(M) & & H^j(M) \oplus H^j(M) \\ (x, y) & \longmapsto & (x-y,\ (h^{-1})^* x - y). \end{array}$$

given by $M = {}_1M \hookrightarrow M_1 \times I$

$$\begin{array}{c} \| \\ M_2 \times I \hookrightarrow M_2 \times I \hookleftarrow {}_2M \end{array}$$

where

$$\begin{array}{cccc} i_1^* : & H^j(M_1 \times I) & \longrightarrow & H^j({}_1M \sqcup {}_2M) \\ & \| & & \| \\ & H^j(M) & & H^j(M) \oplus H^j(M) \\ & x & \longmapsto & (x, (h^{-1})^* x). \end{array}$$

$$\begin{array}{cccc} i_2^* : & H^j(M_2 \times I) & \longrightarrow & H^j({}_1M \sqcup {}_2M) \\ & \| & & \| \\ & H^j(M) & & H^j(M) \oplus H^j(M) \\ & y & \longmapsto & (y, y) \end{array}$$

$h^* x = x$

$$\Rightarrow \ker \ell^j = \{ (x, x) \in H^j(M) \oplus H^j(M) \mid (h^{-1})^* x = x \}$$

$$\cong \ker((h^{-1})^* - 1 : H^j(M) \circlearrowleft) \cong \ker(h^* - 1 : H^j(M) \circlearrowleft)$$

Now $h^* : H^j(M) \circlearrowleft$ is adjoint of $(h^{-1})^* : H^{4k-1-j}(M) \circlearrowleft$

i.e. $\langle h^* x \cup y, [M] \rangle = \langle x \cup (h^{-1})^* y, [M] \rangle$.

$\Rightarrow h^*-1 : H^j(M) \circlearrowleft$ is adjoint of $(h^{-1})^*-1 : H^{4k-1-j}(M)\circlearrowleft$

$\Rightarrow \dim\ker(h^*-1 : H^j(M)\circlearrowleft) = \dim\ker((h^{-1})^*-1 : H^{4k-1-j}(M)\circlearrowleft)$.

$\Rightarrow \dim\ker \ell^j = \dim\ker \ell^{4k-1-j}$

$\overset{j=2k}{\Longrightarrow} \dim\ker \ell^{2k-1} = \dim\ker \ell^{2k}$ $\square$

$\sigma(Q)=\sigma(W)$ $\quad$ $p_{i_1}\cdots p_{i_e}[Q]=0$ for $e>1$

Rk: Facts 1 and 2 were proved by [B65] and were used to prove

Th ([B65]). Let M^{4k-1} be a closed stably parallelizable mfd with $H^1(M;\mathbb{Z}_2)=0$, $k>1$. Let T be a homotopy $(4k-1)$-sphere which bounds a parallelizable mfd W. If $T\# M\equiv M$, then

(i) If k odd, then $T\equiv S^{4k-1}$

(ii) If K even, then $2T = T\# T\equiv S^{4k-1}$.

Rk: There is a modified version of Prop 3.23. as below, which is similarly proved and is used later.

Prop 3.23': For each $k\geq 2$, $\exists$ prime $p=p(k)$ $(\neq 2)$. s.t. the following (1)(2) hold:

(1) $p\mid t_k$

(2) If $[\Sigma]\in\Theta_{4k-1}(\partial\pi)$ and $\Sigma=\partial W$ (W parallel.) M: oriented closed mfd, and $\exists F: M\xrightarrow{\equiv} M\#\Sigma$ s.t. all decomposable Pontrjagin numbers of $\underset{(M\times[0,1]\cup W)_F}{\overset{\shortparallel}{Q}}$ are zero, then $\Sigma\in\langle p\rangle$.

Now turn to

Th 3.15 (Farrell-Sorcar '17)
For every $n=4k-2$, where $k \in \mathbb{Z}_{\geq 2}$, $\exists$ complex

hyp. mfd M^n s.t.

$$\pi_1 T^{<0}(M) \xrightarrow{F_*} \pi_1 T(M)$$

is nontrivial.

Recall complex hyp. mfd.

Begin with real hyp. mfd.

Consider

$$\langle \cdot , \cdot \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

$$SO(n,1) = \{ A \in SL(n+1, \mathbb{R}) \mid \forall x, y \in \mathbb{R}^{n+1}, \langle Ax, Ay \rangle = \langle x, y \rangle \}.$$

$G = SO_0(n,1) \triangleq$ identify component of $SO(n,1)$.

The maximal compact group K of G is
$SO(n) \times SO(1) = SO(n)$

$$G/K = SO_0(n,1)/SO(n) = \mathbb{H}^n \text{ (real } n\text{-dim hyp. space)}$$

Rk: G/K has natural Rieman. metric, coming from killing form on Lie grp G.

If Γ is a torsion-free, discrete subgrp of $G = SO_0(n,1)$, then the locally symmetric space

$$X = \Gamma \backslash G/K = \Gamma \backslash SO_0(n,1)/SO(n) \text{ (with natural metric)}$$

is a real hyp. mfd.

is a real hyp. mfd.

cf. [H, P.453], [Oo1, §3].

Consider
$$\langle\cdot,\cdot\rangle: \mathbb{C}^{n+1}\times\mathbb{C}^{n+1} \longrightarrow \mathbb{C}$$
$$(x,y) \longmapsto \sum_{i=1}^{n} x_i\bar{y}_i - x_{n+1}\bar{y}_{n+1}.$$
→ nondeg. indefinite Hermitian form.

{行列式为1的n+1阶复矩阵}

$$G = SU(n,1) = \{A\in SL(n+1,\mathbb{C}) \mid \langle Ax, Ay\rangle = \langle x,y\rangle, \forall x,y\in\mathbb{C}^{n+1}\}$$

The maximal compact subgrp K of G is $S(U(n)\times U(1))$

$$= \left\{ \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} \middle| \begin{array}{l} A_1\in U(n),\ A_2\in U(1) \\ \text{and } \det A_1 \det A_2 = 1 \end{array} \right\}.$$

$$G/K = SU(n,1)\big/S(U(n)\times U(1))$$ (simply-conn.).

is complex/Hermitian n-dim hyp. space

If Γ is a torsion-free discrete subgrp of G, then locally symmetric space

$$\Gamma\backslash G/K = \Gamma\backslash SU(n,1)\big/S(U(n)\times U(1))$$ (with natural metric)

is a complex hyp. mfd.

Prop 3.24 Given $r\geq 1$ and a closed complex hyp. mfd (N^n, g_c) of dim n, there $\exists$ finite cover $\hat{N}^n$ of N^n and a neg. curved metric g_s on $\hat{N}$ s.t. $(\hat{N}, g_s)$ contains a hyp. geodesic ball $B(p, 2r)$ and $g_s = \hat{g}_c$ on $\hat{N}\setminus B(p, 9r^2)$,

where $\hat{g}_{\mathbb{C}}$ is the induced complex hyp. metric on $\hat{N}$.
cf. [FS17] [FJ94].

Rk: It is unknown if $[g_s] = [\hat{g}_{\mathbb{C}}] \in \pi_0 \mathcal{T}^{<0}(\hat{N})$.

Rk: [FJ94] proved that for each $m = 4k+1$, $k \in \mathbb{Z}_{\geq 1}$, and $\varepsilon > 0$, there $\exists$ a pair of compact smooth mfds M^{2m} and N^{2m} s.t. (a)(b)(c) hold:
(a) $M \approx N$ but $M \not\cong N$
(b) M is a complex hyp. mfd
(c) N supports a neg. curved metric with sec. curvature $\in [-4-\varepsilon, -1+\varepsilon]$

Note that in this result ε cannot be 0 by [H91] [YZ91]: They show that

if $M \simeq N$: compact smooth mfd s.t. M^{2m} is complex hyp. mfd and N^{2m} supports neg. curved metric with sec. $\in [-4, -1]$, then M^{2m} is isometric to N^{2m}.

• Some digression on dual symmetric space cf. [O01]

Example

$\mathbb{H}^n \xleftrightarrow{\text{dual}} S^n$ $\quad\quad$ $(\overset{X}{\mathbb{H}^n/\Gamma} \xleftrightarrow{\text{dual}} \overset{X_u}{S^n})$ $\quad$ Γ: torsion-free

Example

$$\mathbb{H}^n \xleftrightarrow{\text{dual}} S^n \qquad\qquad \overset{X}{(\mathbb{H}^n/\Gamma} \xleftrightarrow{\text{dual}} \overset{X_u.}{S^n)}$$

$$\mathbb{C}\mathbb{H}^n \xleftrightarrow{\text{dual}} \mathbb{C}P^n \qquad\qquad (\mathbb{C}\mathbb{H}^n/\Gamma \xleftrightarrow{\text{dual}} \mathbb{C}P^n)$$

complex n-dim hyp. space

Γ: torsion-free discrete, cocompact

By [M62],
$$H^*(\mathbb{H}^n/\Gamma;\mathbb{R}) \cong H^*(S^n;\mathbb{R})$$
$$H^*(\mathbb{C}\mathbb{H}^n/\Gamma;\mathbb{R}) \cong H^*(\mathbb{C}P^n;\mathbb{R})$$
$\Big\}$ up to certain degree.

In g al, let G be semisimple linear algebraic group defined over $\mathbb{Q}$; $G(\mathbb{R})$: the group of real points of G, is a Lie group. We will denote $G(\mathbb{R})$ by G.

K: maximal compact subgrp of G.
G_c: complexification of G
G_u: maximal compact subgrp of G_c

Def: G/K and $X_u = G_u/K$ are called dual symmetric space of noncompact type and compact type, respectively. Let Γ be a torsion-free, discrete subgrp of G of finite covolume.

$X = \Gamma \backslash G/K$ is a locally symmetric space of noncompact type.

Refer to X and X_u as dual symmetric spaces.

sec ≤ 0 G/K, $X = \Gamma\backslash G/K$, $X_u = G_u/K$. sec ≥ 0 → cf. [H, chap V §3]

have natural Riem. metrics, coming from the Killing form on G.

Example

1) $G = SO_0(n,1)$

$$G/K = SO_0(n,1)/SO(n) = \mathbb{H}^n$$

$$X = \Gamma\backslash G/K = \mathbb{H}^n/\Gamma \quad \text{— real hyp. mfd}$$

$$X_u = G_u/K = SO(n+1)/SO(n) = S^n. \quad (G_c = SO(n+1,\mathbb{C}))$$

2) $G = SU(n,1)$

$$G/K = SU(n,1)/S(U(n)\times U(1)) = \mathbb{C}\mathbb{H}^n.$$

dual

$$X = \Gamma\backslash G/K = \mathbb{C}\mathbb{H}^n/\Gamma \quad \text{complex hyp. mfd.}$$

$$X_u = G_u/K = SU(n+1)/S(U(n)\times U(1)) = \mathbb{C}P^n. \quad (G_c = SL(n+1,\mathbb{C}))$$

[M62] defines a homomorphism $j^*: H^*(X_u;\mathbb{R}) \longrightarrow H^*(X;\mathbb{R})$

Th 3.25 [M62] If X is compact, then
(1) j^* is injective
(2) j^q is surjective for $q \le m(g)$
where $m(g)$ is explicitly defined in terms of Lie algebra g of G.

Rk: [G71] [B74] extend Th 3.25 to the case where

- X is noncompact but of finite volume
- Γ has to be arithmetric grp. $\to$ eg. $SL(n,\mathbb{Z})$. $\to$ cf. eg. [B. P217. 5].

Example Borel [B74] compute rational Alg. K-theory of $\mathbb{Z}$: for $i \geq 2$

$$K_i(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 1 & i \equiv 1 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

Recall $K_i(\mathbb{Z}) = \pi_i(BGL(\mathbb{Z})^+)$ for $i \geq 1$.
where $BG^+ \triangleq BGL(\mathbb{Z})^+$ is a CW-complex
s.t. $\exists\, \sigma : BG \longrightarrow BG^+$ satisfying (1) (2):

(1) $\ker(\sigma_\# : \pi_1 BG \longrightarrow \pi_1 BG^+) = [G, G]$ note: $\pi_1 BG = G$

(2) $\sigma_* : H_*(BG) \longrightarrow H_*(BG^+)$

$$\begin{aligned}
K_i(\mathbb{Z}) \otimes \mathbb{R} &= \pi_i(BG^+) \otimes \mathbb{R} \\
&\uparrow \wr \\
H_*(BG^+; \mathbb{R}) & \\
\| & \\
H_*(BG; \mathbb{R}) &= H_*(GL(\mathbb{Z}); \mathbb{R}) \\
&\qquad \uparrow \wr \\
H^*(SL(\mathbb{Z}); \mathbb{R}) &= H^*(GL(\mathbb{Z}); \mathbb{R}) \\
\| & \\
H^*(\varinjlim_n SL(n, \mathbb{Z}); \mathbb{R}). &
\end{aligned}$$

Now consider $G = SL(n, \mathbb{R})$.

$K = SO(n)$, $G_c = SL(n, \mathbb{C})$, $G_u = SU(n)$

$$X_{u,n} = G_u/K = SU(n)/SO(n).$$

$$\Gamma_n \overset{\text{finite index}}{\triangleleft} SL(n,\mathbb{Z}) < G$$

torsion

discrete
not torsionfree

Existence of Γ_n is guaranteed by Selberg's lemma ([Selberg, 1961])

$$X_n = \Gamma_n \backslash G/K = \Gamma_n \backslash SL(n,\mathbb{R})/SO(n)$$

$H^*(X_{u,n};\mathbb{R}) \xrightarrow{\cong} H^*(X_n;\mathbb{R})$ in some range.

$n >> *$

$$\Downarrow$$

$$H^*(\varinjlim_n X_{u,n};\mathbb{R}) \xrightarrow{\cong} H^*(\varinjlim_n X_n;\mathbb{R})$$

compute by [Cartan, 1961]

$G/K \approx \mathbb{R}^{\dim G - \dim K}$

$\Downarrow$

$X_n = B\Gamma_n$

$$\| \\ H^*(\varinjlim_n \Gamma_n;\mathbb{R}) \\ \| \\ H^*(\varinjlim_n SL(n,\mathbb{Z});\mathbb{R})$$

Th 3.26 [Oo1] For dual symmetric space X and X_u,

$X = \Gamma\backslash G/K$, $X_u = G_u/K$

$\exists$ a finite cover X' of X and a tangential map

$$k: X' \longrightarrow X_u.$$

i.e. $k^* TX_u = TX'$

i.e. $k^* T X_u = TX'$

<u>Cor</u> Let X^n be a closed real hyp. mfd. Then $\exists$ finite cover $\hat{X}$ of X s.t.

$$T\hat{X} \oplus \varepsilon^1 = \varepsilon^{n+1}.$$

<u>Pf</u>: By Okun's result, $\exists$ finite cover $\hat{X}$ of X and cont. map

$$k : \hat{X} \longrightarrow X_u = S^n$$

s.t. $k^* T S^n = T\hat{X}$

$$\xrightarrow{TS^n \oplus \varepsilon^1 = \varepsilon^{n+1}} T\hat{X} \oplus \varepsilon^1 = \varepsilon^{n+1}.$$

$\square$.