Gaussian Multiplicative Chaos

Random bounded analytic function by random measure

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THU-PKU-BNU Probability Webinar

Outline

1. Basic facts about Gaussian Multiplicative Chaos measures

2. Basic facts about Clark measures

3. Basic facts about GMC measures as Clark measures

Warning: Clark measure is also known as spectral measure, Alesandrov measure, or Alesandrov-Clark measure, and also appears in works of Simon-Wolff on Anderson localization.

Basic facts about Gaussian variables

Consider a standard Gaussian variable N and its normalized exponential

$$E(t) = e^{tN-\frac{t^2}{2}}, \quad 1 \ll t.$$

1. The probability that $E(t)$ is about 1 is about $e^{-t^2/8}$.
   ☕ It is roughly the probability that $N \simeq t/2 \cdot \mathrm{var}(N)$.
2. We have $\mathbb{E}[E(t) \wedge 1] \simeq e^{-t^2/8}$, and $\mathbb{E}[E(t)^{1/2}] = e^{-t^2/8}$.
   ☕ So really the $\{E(t) \simeq 1\}$ region contributes to the first expectation.
3. The expectation of $E(t)$ is 1, although $E(t)$ is usually very small.
   ☕ The $\{N \simeq t \cdot \mathrm{var}(N)\}$ region contributes to this expectation.

Kahane 85

"$e^{\gamma X}$"

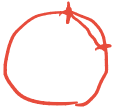
X log-corr Gaussian field

Gaussian Multiplicative Chaos

Log-correlated Gaussian fields in 1d

As a formal Gaussian process (in fact, random generalized function):

$$\mathbb{E}\left[X(z)X(z')\right] = \ln \frac{1}{|z-z'|} + O(|z-z'|).$$



Examples:

$$\mathbb{E}\left[X(e^{i\theta})X(e^{i\theta'})\right] = \ln \frac{1}{|e^{i\theta}-e^{i\theta'}|},$$ "canonical" on the unit circle

$$\mathbb{E}\left[X(z)X(z')\right] = \ln \frac{1}{|z-z'|},$$ "canonical" on the unit interval

Basic facts about log-correlated Gaussian fields: the Fourier viewpoint

Brownian bridge as a random Fourier series on the unit interval $t \in [0, 1]$:

$$B_t = \sum_{n=1}^{\infty} \frac{\sqrt{2} Z_n}{\pi n} \sin(\pi n t).$$

$$\mathbb{E}[B_s B_t] = s(1 - t).$$

Canonical log-correlated Gaussian field on the unit circle $\theta \in [0, 2\pi]$:

$$X(\theta) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (A_n \cos(n\theta) + B_n \sin(n\theta)).$$

$$\mathbb{E}[X(\theta) X(\theta')] = -\ln|e^{i\theta} - e^{i\theta'}|.$$

In other words: Hilbert space structure and Karhunen-Loève expansion!

Basic facts about log-correlated Gaussian fields: the multifractal viewpoint

Brownian bridge on $[0,1]$:

$$\mathbb{E}[B_s B_t] = s(1-t), \quad s,t \in [0,1].$$

Brownian bridge on $[0,1/2]$:

$$\mathbb{E}[B_{s/2} B_{t/2}] = s/2(1/2 - t/2) = 4\mathbb{E}[B_s B_t].$$

Canonical log-corr field on $[0,1]$:

$$\mathbb{E}[X(z)X(z')] = -\ln|z-z'|, \quad z,z' \in [0,1].$$

Exact-scaling log-corr field on $[0,1/2]$:

$$\mathbb{E}[X(z/2)X(z'/2)] = \mathbb{E}[X(z)X(z')] + \ln 2.$$

$+\sqrt{\ln 2}\, N$

Comparing different log-correlated fields: Kahane's convexity inequality!

Artists in Residence: Gaussian Free Field in two dimension

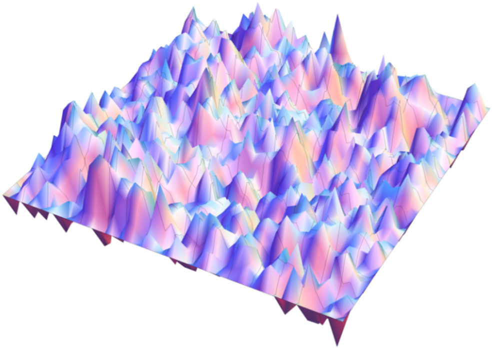


Figure 1: Simulation by Jacopo Borga.

Gaussian multiplicative chaos measure in 1d

Let $\gamma \in (0, \sqrt{2})$. [Kahane 85′]: define a random measure $\mu(T) = \int_T \underline{e^{\gamma X(\theta)}}\, d\theta$, where X is a log-correlated field (in any dimension).

"$e^{\partial X}$"

1. Regularize the field: $\boxed{X_\epsilon(\theta)}$ as "average" of X on $\underline{[\theta - \epsilon, \theta + \epsilon]}$.
   ☕ $\mathbb{E}[X_\epsilon(\theta)^2] \simeq -\ln \epsilon$.

2. Define random measures $\underline{\mu_\epsilon(T)} = \int_T \underline{e^{\gamma X_\epsilon(\theta) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(\theta)^2]}}\, d\theta \simeq \epsilon^{\frac{\gamma^2}{2}} \int_T e^{\gamma X_\epsilon(\theta)}\, d\theta$.
   ☕ $\mathbb{E}[\mu_\epsilon(T)] = |T|$.

3. Show weak convergence of measures $\underline{\mu_\epsilon \to \mu}$. GMC
   ☕ $\mathbb{E}[\mu_\epsilon(T)^p]$ uniformly bounded for any $p < \frac{2}{\gamma^2}$ (also negative p).

Intuition for following this talk

Let $\gamma \in (0, \sqrt{2})$. [Kahane 85']: define a random measure $\mu(T) = \int_T e^{\gamma X(\theta)} d\theta$.

Q: What is the behavior of $\mu([0,r])$ compared to $\mu([0,1])$ for small r?

1. Scaling of the field X: $\{X(rz)\}_{z\in[0,1]} = \{\boxed{\sqrt{-\ln r}N} + X(z)\}_{z\in[0,1]}$ in law.
   ☕ $\mathbb{E}[X(rz)X(rz')] = -\ln(r|z-z'|) = -\ln r + \mathbb{E}[X(z)X(z')]$.
2. Scaling of the measure μ: $\underset{\text{GMC}}{\mu([0,r])} = \boxed{e^{\gamma\sqrt{-\ln r}N - \frac{\gamma^2}{2}\ln\frac{1}{r}}} \cdot r \cdot \mu([0,1])$ in law.
   ☕ $\mathbb{E}[: e^{\gamma\sqrt{-\ln r}N} :] = 1$.

Q: Geometric interpretation of N?

- Intuition: on the interval $[\theta - r, \theta + r]$, $N \simeq X_r(\theta)$.
  ☕ The underlying Gaussian fields are almost independent.

Basic facts about Gaussian Multiplicative Chaos

Use the intuition: $\frac{1}{r}\mu([0,r]) = e^{\gamma\sqrt{-\ln r}N - \frac{\gamma^2}{2}\ln\frac{1}{r}} \cancel{\mu([0,1])}$; imagine $\mu([0,1]) = O(1)$.

1. Large deviation of the measure: $\mathbb{P}[\frac{1}{r}\mu([0,r]) \simeq 1] \simeq r^{\frac{\gamma^2}{8}}$ for small r.
   ☕ $\mathbb{P}[e^{\gamma\sqrt{-\ln r}N - \frac{\gamma^2}{2}\ln\frac{1}{r}} \simeq 1] \simeq \mathbb{P}[N \simeq \frac{\gamma}{2}\sqrt{-\ln r}]$ (so-called $\frac{\gamma}{2}$-thick point).
2. Support of the measure μ: supported on $\boxed{\gamma\text{-thick points}}$, $X_r(\theta) \simeq \gamma\sqrt{-\ln r}$.
   ☕ Contribution to $\mu([0,1])$ from α-thick points $\simeq r^{\frac{\alpha^2}{2}} \cdot r^{-\alpha\gamma + \frac{\gamma^2}{2}}$.
3. Fractal dimension of α-thick points: a.s. $\dim_{\mathcal{H}}\{\theta \text{ is } \alpha\text{-thick}\} = \boxed{1 - \frac{\alpha^2}{2}}$.
   ☕ $\mathbb{P}[X_{1/N}(\theta) \simeq \alpha\sqrt{\ln N}] \simeq N^{-\frac{\alpha^2}{2}}$ for large N (see Hu-Miller-Peres).

The GMC measure is almost surely singular to Lebesgue!

Artists in Residence: GMC with $\gamma = 1.6$ in two dimension

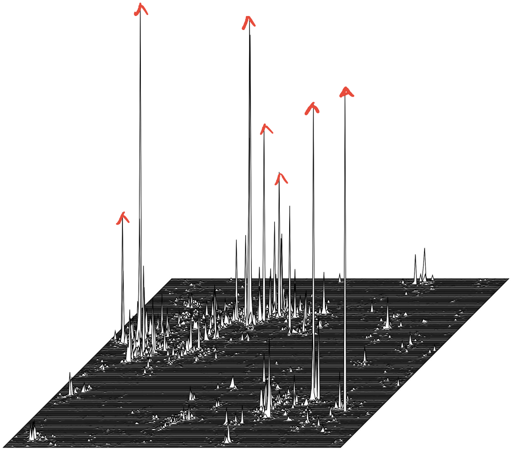


Figure 2: Simulation by Michel Pain.

The extended Seiberg bound in Liouville conformal field theory

Intuition: a multifractal measure integrates more singularity than Lebesgue.

Statement: consider the mass of GMC on $[-1,1]$ with log-singularity at 0. Then

$$M_{p,s}([-1,1]) = \mathbb{E}\left[\left(\int_{-1}^{1} |\theta|^{-s}\, d\mu(\theta)\right)^{p}\right] < \infty$$

for $0 < p < 1$ if and only if $s < 1 + \frac{\gamma^2}{2}(1-p)$.

☕ By scaling, $M_{p,s}([-1/2,1/2]) = 2^{sp-p(1+\gamma^2/2)+p^2\gamma^2/2} \cdot M_{p,s}([-1,1])$.

Use in this talk: control of non-compact regularizations.

Two messages

Remember these for the rest of the talk!

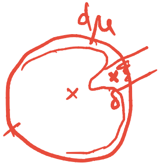
1. GFF (and therefore GMC) has a random Fourier representation!
2. Average of GMC on small intervals behaves like the exponential of a Gaussian.

Clark measure and analytic function

Basic facts about the harmonic extension of a measure

Take a non-negative measure $d\mu(\theta)$ on the unit circle. We can extend it to a harmonic function $x(z)$ inside the unit disc via the Poisson kernel:

$$x(z) = \int_0^{2\pi} P_z(e^{i\theta})d\mu(\theta), \quad P_z(e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2}.$$

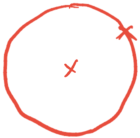


1. Poisson kernel is a non-compact regularization. Think $x(z) \simeq X_{1-|z|}(z/|z|)$, i.e. with $\theta = z/|z|$, the average of μ on $[\theta - (1 - |z|), \theta + (1 - |z|)]$.
2. Lebesgue's decomposition: write $\mu = \sigma + \mu^a dl$ with σ singular and μ^a the density. As $r \to 1$, $\mu(r\theta) \to \mu^a(\theta)$ for dl-a.e. θ and $\mu(r\theta) \to \infty$ for σ-a.e. θ.

Basic facts about the holomorphic extension of a measure

Consider the harmonic conjugate $y(z)$ of the harmonic extension $x(z)$ with $y(0) = 0$. Then $x + iy$ is holomorphic in the unit disc, i.e.

$$x(z) + i\boxed{y(z)} = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$



1. The function y is the Hilbert transform of the function x on the boundary.
2. A theorem of Riesz roughly says that $|y(z)|$ cannot be much larger than $x(z)$.
3. Explicitly, $y(z)$ can be written with the kernel $Q_z(e^{i\theta}) = \frac{-2|z|\sin(\theta - \arg(z))}{|z - e^{i\theta}|^2}$.

Basic facts about Clark measures

Given an analytic self map $\varphi : \mathbb{D} \to \mathbb{D}$ of the unit disc of the complex plane, for each $|\alpha| = 1$, the measure $\nu_\alpha = \nu_{\varphi,\alpha}$ is defined via

$$\operatorname{Re}\left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}\right) = \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \nu_\alpha(d\theta), \qquad z \in \mathbb{D}.$$

☕ The function $\operatorname{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right)$ is harmonic and non-negative.

$\{\nu_\alpha\}_{\alpha \in \mathbb{T}}$

The measure ν_α, or especially its singular part, describes how strongly and where on the boundary the function φ takes the value α.

Examples of Clark measures

1. Elementary example: $\varphi(z) = z^n$. Then ν_α is n point-masses, each of mass $1/n$, located at the n roots of unity of α.

   $\varphi(\zeta) = \alpha$

2. Atomic inner function: $\varphi(z) = \exp\left(\frac{z+1}{z-1}\right)$. Then ν_α is discrete, supported on $\{\zeta\ ;\ \varphi(\zeta) = \alpha\}$ and each mass equals $|\zeta - 1|^2/2$.

inner function

Fact! The analytic function φ has non-tangential limit $|\varphi(e^{i\theta})| = 1$ a.e. if and only if ν_α is singular for some α (or for all α).

Let $\nu_{\varphi,\alpha=1}$ be the GMC measure and study the random inner function φ.

singular

Decomposition of inner functions

An inner function is a bounded analytic function on $\mathbb{D}$ with $|f(e^{i\theta})| = 1$ a.e.

1. Möbius map: $\alpha_w(z) = \frac{w-z}{1-\overline{w}z}$, $z \in \mathbb{D}$.
   ☕ Only zero at $w = z$.
2. Blaschke product: B product of Möbius maps (and maybe some angle).
   ☕ Determined by its zeroes; can be used to eliminate zeroes.
3. Singular factor: $S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\nu(\theta)\right)$, $d\nu \perp d\theta$, $d\nu \geq 0$.
   ☕ Example: $S(z) = \exp\left(\frac{z+1}{z-1}\right)$, no zeroes in $\mathbb{D}$.

Canonical Factorization Theorem: every inner function f is

$$f = e^{ic} B(z) S(z), \quad c \in \mathbb{R}.$$

GMC measures as Clark measures

Frostman's lemma for inner functions

Philosophy: the singular factor is unstable under most conformal maps.

Frostman shift: for any holomorphic self-map φ of $\mathbb{D}$ and any $w \in \mathbb{D}$, define

$$\varphi_w = \alpha_w \circ \varphi, \qquad \alpha_w(z) = \frac{w - z}{1 - \overline{w} z}.$$

Frostman's lemma: if φ is inner, then φ_w is a Blaschke product for quasi-every $w \in \mathbb{D}$, i.e. except a set of log-capacity zero.

A question of Hedenmalm and Poltoratski

Question: does the same holds for random Clark measure by GMC?

More precisely, pick a GMC μ on the unit circle and define it as the Clark measure at $\alpha = 1$. Use this to define a random holomorphic function φ. The function φ is a.s. inner since μ is a.s. singular w.r.t to Lebesgue. But is it so that φ is a.s. a Blaschke product, i.e. without singular inner factor?

Anwser (H.-Saksman): Yes! The measure μ is characterized by pure-point data.

Ideas of proof: perturbation of Fourier coefficient

1. By simple inequalities, it suffices to show that the imaginary part $y(z)$ has uniformly bounded negative moments, i.e. for some $0 < p < 1$,

$$\sup_{r\in[0,1)} \mathbb{E}\left[\frac{1}{|y(r)|^p}\right] < \infty, \quad y(r) = \int_0^{2\pi} \frac{-2r\sin(\theta)}{|r-e^{i\theta}|^2} d\mu(\theta).$$

2. Recall $X_c(\theta) = \boxed{B_1 \sin(\theta)} + \widetilde{X}(\theta)$ so $d\mu(\theta) = \exp\left(\gamma B_1 \sin(\theta) - \frac{\gamma^2 \sin^2(\theta)}{2}\right) d\widetilde{\mu}(\theta)$.
   Observe that $\boxed{-\frac{\partial y(r)}{\partial B_1}}$ is of constant sign! Furthermore, lower bound via $\widetilde{\mu}$.
3. Conclude with moment bounds of $\widetilde{\mu}$ since it is also a GMC measure.

A probabilistic criteria for Frostman's lemma: the random inner function φ is a.s. Blaschke product if for some $\epsilon > 0$,

$$\sup_{z\in\mathbb{D}} \mathbb{E}\left[(-\ln|\varphi(z)|)^{1+\epsilon}\right] < \infty.$$

Density of random zeroes

Let $\{z_k\}_{k\geq 1}$ be zeroes of the Blaschke product φ. Then

$$\sum_{k\geq 1}(1-|z_k|)^{1} < \infty.$$

Question: for the GMC problem, which $0 < \alpha < 1$ do we have almost surely

$$\sum_{k\geq 1}(1-|z_k|)^{\alpha} < \infty.$$

Anwser (H.-Saksman): the threshold is $\alpha = 1 - \frac{\gamma^2}{8}$.

Ideas of proof

$e^{\gamma X}$ $\to \gamma X$ $\frac{\gamma}{2}X$ e $p \to 1$

1. By Green's formula

$$\sum_{k\geq 1} u(z_k) = \frac{1}{2\pi}\int_{\mathbb{D}} u(z)\Delta \ln|\varphi(z)|dA(z) = \frac{1}{2\pi}\int_{\mathbb{D}} \ln|\varphi(z)|\Delta u(z)dA(z),$$

roughly reduce to proving

$$(1-r)^{\frac{\gamma^2}{8}+\epsilon} \lesssim \int_0^{2\pi} -\ln|\varphi(re^{i\theta})|d\theta \lesssim (1-r)^{\frac{\gamma^2}{8}-\epsilon}, \quad r \to 1^-.$$

2. Use $-\ln|\varphi(z)| = \ln\left(1 + \frac{4\cancel{x}(z)}{(\cancel{x}(z)-1)^2+\cancel{y(z)^2}}\right)$. Notice the singularity at $x(z) = 1$.
3. For the upper bound: show $\mathbb{E}\left[\ln\left(1+\frac{4x(z)}{(\cancel{x}(z)-1)^2}\right)\right] \lesssim (1-|z|)^{\frac{\gamma^2}{8}-\epsilon}$.
   ☕ Perturbative method similar to the probabilistic Frostman's lemma.
4. For the lower bound: multifractal analysis to the level set $\{x \sim 1, y \lesssim 1\}$.
   ☕ Invent from $e^{\gamma X}$ some random variable supported on $\frac{\gamma}{2}$-thick points.

"$e^{\frac{\gamma}{2}X}$"

Upper bound via the extended Seiberg bound

Recall our goal: $\mathbb{E}\left[\ln\left(1+\frac{4x(z)}{(x(z)-1)^2}\right)\right] \lesssim (1-|z|)^{\frac{\gamma^2}{8}-\epsilon}$.

1. Extend the proof of probabilistic Frostman: use rank-two perturbation to "swipe through" the singularity at $x = 1$. Left with roughly $\mathbb{E}\left[x(z) \wedge 1\right]$.
2. With the Poisson kernel, $x(r) = \int_0^{2\pi} \frac{1-r^2}{|r-e^{i\theta}|^2} d\mu(\theta)$ is roughly the mass of a GMC with singularity at 1. Compare this singularity with the extended Seiberg bound to optimize $\mathbb{E}[x(z)^p]$ for $0 < p < 1$.
3. Optimize the parameters (with $p = \frac{1}{2}$) to bound $\mathbb{E}\left[x(z) \wedge 1\right] \leq \mathbb{E}[x(z)^p]$.

☕ The last bound is effective around $x(z) \sim 1$, which has probability $\simeq (1-|z|)^{\frac{\gamma^2}{8}}$.

Lower bound via multifractal analysis of level sets of GMC

Recall our goal: $(1-r)^{\frac{\gamma^2}{8}+\epsilon} \lesssim \int_0^{2\pi} -\ln|\varphi(re^{i\theta})|d\theta = \int_0^{2\pi} \ln\left(1+\frac{4x}{(x-1)^2+y^2}\right)d\theta.$

1. On the level set $\{x \sim 1, y \leq 1\}$, $-\ln|\varphi(re^{i\theta})|$ is bounded below by positive constant. It suffices to show $|\{x \sim 1, y \leq 1\}| \simeq (1-r)^{\frac{\gamma^2}{8}}$ in $r\mathbb{T}$.
2. Probabilistic analogue of Riesz theorem: reduce $|\{x \sim 1, y \leq 1\}|$ to $|\{x \sim 1\}|$.
3. Use multifractal analysis: heuristically, $x \sim 1$ is the set of $\frac{\gamma}{2}$-thick points.

   ☕ Recall that $\mathbb{P}[x(z) \simeq 1] \simeq (1-|z|)^{\frac{\gamma^2}{8}}$.
4. Trick: invent a variable defined by $e^{\gamma X}$ that scales like $e^{\frac{\gamma}{2}X}$!

$p\gamma$

$$M_{p,\epsilon}(I) = \int_I \epsilon^{-(p-p^2)\gamma^2}\left(\frac{1}{2\epsilon}\mu([\theta-\epsilon,\theta+\epsilon])\right)^p d\theta, \quad I \in \mathbb{T}.$$

In some sense, $\lim_{\epsilon\to 0} M_{p,\epsilon}(I)$ behaves like $\lim_{r\to 1^-}(1-r)^{-\frac{\gamma^2}{8}}|\{x \sim 1\}|$.