Lecture No 18 May 10, 2022 (Tue)

Recall Cameron-Martin-Girsanov-Maruyama's formula:

- $(\Omega, \mathcal{F}, \mathcal{P})$, $(\mathcal{F}_t)_{t\geq 0}$: Probability space and its filtration
- $B = (B_t(\omega))_{t\in[0,1]}$: 1-dimensional $(\mathcal{F}_t)$-Brownian motion such that $B_0 = 0$.
- $\eta_t = \eta(t, \omega)$, $t \in [0, 1]$: $(\mathcal{F}_t)$-adapted stochastic process and absolutely continuous in t s.t. $\eta_0 = 0$ and

$$\sup_{\omega\in\Omega} \int_0^1 \dot{\eta}_t(\omega)^2 \, dt < \infty.$$

[Theorem 18.1] One can define a probability measure Q on $(\Omega, \mathcal{F})$ by

$$\frac{dQ}{dP}(\omega) = \exp\left\{\int_0^1 \dot{\eta}_t dB_t - \frac{1}{2}\int_0^1 \dot{\eta}_t^2 \, dt\right\},$$

and under Q,

$$\hat{B}_t := B_t - \eta_t, \quad t \in [0, 1]$$

is a Brownian motion. □

To give a proof of Theorem 18.1, we prepare a lemma.

[Lemma 18.3] Assume that an $(\mathcal{F}_t)$-adapted continuous process $X = (X_t)_{t \geq 0}$ satisfies $X_0 = 0$ (a.s.) and

$$\varphi(X_t) - \frac{1}{2}\int_0^t \varphi''(X_s)\, ds : \ (\mathcal{F}_t)\text{-martingale}$$

for $^\forall \varphi \in C_b^2(\mathbb{R})$. Then, X is an $(\mathcal{F}_t)$-Brownian motion. □

⚇ Take $\varphi(x) = e^{i\xi x}, \xi \in \mathbb{R}$ (or its real part and imaginary part separately). Then, by the condition, $e^{i\xi X_t} + \frac{\xi^2}{2}\int_0^t e^{i\xi X_s}\, ds$ is an $(\mathcal{F}_t)$-martingale. Therefore, for any $\mathcal{F}_s$-measurable and bounded function Φ on Ω, we have

$$\tfrac{d}{dt} E\left[e^{i\xi X_t}\Phi\right] = -\tfrac{\xi^2}{2} E\left[e^{i\xi X_t}\Phi\right], \quad t \geq s > 0$$

One can easily solve this ODE and obtain

$$E\left[e^{i\xi X_t}\Phi\right] = E\left[e^{i\xi X_s}\Phi\right] e^{-\frac{\xi^2}{2}(t-s)}$$

Take $e^{-i\xi X_s}\Phi$ for Φ and we obtain $E[e^{i\xi(X_t - X_s)}\Phi] = e^{-\frac{\xi^2}{2}(t-s)}E[\Phi]$. This implies that $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s$, $X_t - X_s \stackrel{\text{law}}{=} N(0, t-s)$. Thus, the conclusion is shown. □

[Remark] (1) If X is a Brownian motion, the property stated in Lemma 18.3 holds. Recall Proposition 16.1 with $\mathcal{L}_s = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ for the Brownian motion.

(2) Lemma 18.3 claims that the converse is also true, or one can say that the solution of the martingale problem for $\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ is unique and it is the Wiener measure.

(3) By the characterization of the Brownian motion by martingale (Lévy's theorem), for X to be a Brownian motion, it is actually enough to check the above martingale property only for $\varphi(x) = x, x^2$. In other words, if X_t is a square integrable martingale and $X_t^2 - t$ is also a martingale (note that $\frac{1}{2}(x^2)'' = 1$), then X_t is an $(\mathcal{F}_t)$-Brownian motion. □

[Proof of Theorem 18.1] [Step 1] Set

$$M_t := \exp\left\{ \int_0^t \dot{\eta}_s dB_s - \frac{1}{2}\int_0^t \dot{\eta}_s^2\, ds \right\}, \quad t \in [0,1]$$

under the probability measure P (i.e. we define $\int_0^t \dot{\eta}_s dB_s$ as a stochastic integral under P). Then, M_t is an $(\mathcal{F}_t)$-martingale.

∵ By Itô's formula,

$$\begin{aligned} dM_t &= M_t \left\{ \dot{\eta}_t\, dB_t - \tfrac{1}{2}\dot{\eta}_t^2\, dt \right\} + \tfrac{1}{2} M_t \dot{\eta}_t^2\, dt \\ &= M_t \dot{\eta}_t\, dB_t. \end{aligned}$$

In particular, M_t is written as a stochastic integral. Therefore, it is a martingale. □

The above proof is a little rough, since $f(x) = e^x \notin C_b^2$ so that we should be careful to apply Itô's formula. → We need a cutoff argument. (Define $\varphi_n \in C_b^2(\mathbb{R})$ s.t. $\varphi_n(x) = e^x, |x| \le n$ and $\sigma_n := \inf\{t > 0; |M_t| > n\}$. Show $\sigma_n \nearrow \infty$ (a.s.) and $M_{t\wedge\sigma_n}$ is uniformly integrable martingale by the assumption $\sup_{\omega\in\Omega} \int_0^1 \dot{\eta}_t(\omega)^2\, dt < \infty$.) This shows M_t is square integrable.

Since M_t is a martingale, the measure Q on $(\Omega, \mathcal{F})$ defined by $\frac{dQ}{dP} = M_1$ (i.e. $dQ = M_1 dP$) is a probability measure on Ω.

$\because$ The σ-additivity of Q is clear. It is a probability measure, since we have

$$Q(\Omega) = \int_\Omega M_1 dP = E^P[M_1] \underset{M:\text{martingale}}{=} E^P[M_0] \underset{M_0=1}{=} 1. \qquad \square$$

[Step 2] Next, we show that $\hat{B}_t := B_t - \eta_t$ is a Brownian motion under Q. To this end, by Lemma 18.3, it is enough to show that

$$\varphi(\hat{B}_t) - \frac{1}{2}\int_0^t \varphi''(\hat{B}_s)ds$$

is Q-martingale for $^\forall \varphi \in C_b^2(\mathbb{R})$.

Now, under P, noting that $dM_t = M_t \dot{\eta}_t \, dB_t$, by Itô's formula, we have

$$\begin{aligned} d\{M_t \cdot \varphi(\hat{B}_t)\} &= M_t d\{\varphi(\hat{B}_t)\} + \varphi(\hat{B}_t) dM_t + dM_t d\{\varphi(\hat{B}_t)\} \\ &= M_t \varphi'(\hat{B}_t)\{dB_t - \dot{\eta}_t dt\} + \tfrac{1}{2} M_t \varphi''(\hat{B}_t) dt \\ &\qquad + \varphi(\hat{B}_t) M_t \dot{\eta}_t dB_t + M_t \dot{\eta}_t \varphi'(\hat{B}_t) dt \\ &= dN_t + \tfrac{1}{2} M_t \varphi''(\hat{B}_t) dt, \end{aligned}$$

where

$$N_t := \varphi(0) + \int_0^t M_s \left\{ \varphi'(\hat{B}_s) + \dot{\eta}_s \varphi(\hat{B}_s) \right\} dB_s$$

is an $(\mathcal{F}_t)$-martingale under P, since it is a stochastic integral by a Brownian motion (under P). (Note that, since M_t is square integrable, the integrand $\in \mathcal{L}^2_T$.)

Writing this in an integrated form, we have

$$M_t\varphi(\hat{B}_t) = N_t + \frac{1}{2}\int_0^t M_s\varphi''(\hat{B}_s)\,ds.$$

Note that $M_0\varphi(\hat{B}_0) = \varphi(0) = N_0$ holds at $t = 0$. In particular, for $t \geq s \geq 0$, we have

$$M_t\varphi(\hat{B}_t) - M_s\varphi(\hat{B}_s) = N_t - N_s + \frac{1}{2}\int_s^t M_r\varphi''(\hat{B}_r)\,dr. \quad (\star)$$

Using this, we will show for $1 \geq^\forall t \geq^\forall s \geq 0$, ${}^\forall\mathcal{F}_s$-measurable bounded function Φ:

$$E^Q[\varphi(\hat{B}_t)\Phi] = E^Q[\varphi(\hat{B}_s)\Phi] + E^Q\left[\frac{1}{2}\int_s^t \varphi''(\hat{B}_r)\,dr\cdot\Phi\right]. \quad (\star\star)$$

This implies that

$$\varphi(\hat{B}_t) - \frac{1}{2}\int_0^t \varphi''(\hat{B}_s)\,ds \quad \text{is a } (Q,(\mathcal{F}_t))\text{-martingale}$$

and, by Lemma 18.3, this concludes the proof of Theorem 18.1.

Finally, we show $(\star\star)$:

$$
\begin{aligned}
E^Q[\varphi(\hat{B}_t)\Phi] &\underset{(1)}{=} E^P[M_1\varphi(\hat{B}_t)\Phi] \underset{(2)}{=} E^P[M_t\varphi(\hat{B}_t)\Phi] \\
&\underset{(\star)}{=} E^P\left[\left\{M_s\varphi(\hat{B}_s) + N_t - N_s + \tfrac{1}{2}\int_s^t M_r\varphi''(\hat{B}_r)\,dr\right\}\Phi\right] \\
&\underset{(3)}{=} E^P[M_s\varphi(\hat{B}_s)\Phi] + \tfrac{1}{2}\int_s^t E^P[M_r\varphi''(\hat{B}_r)\Phi]\,dr \\
&\underset{(4)}{=} E^Q[\varphi(\hat{B}_s)\Phi] + E^Q\left[\tfrac{1}{2}\int_s^t \varphi''(\hat{B}_r)\,dr\cdot\Phi\right]
\end{aligned}
$$

Here,

- (1) follows by the definition of Q.
- (2): Noting that M_t is $(P,(\mathcal{F}_t))$-martingale, we rewrite
  $E^P[M_1\varphi(\hat{B}_t)\Phi] = E^P[E^P[M_1|\mathcal{F}_t]\varphi(\hat{B}_t)\Phi] = E^P[M_t\varphi(\hat{B}_t)\Phi]$.
- (3) follows since N_t is a P-martingale.
- (4): Similarly to (2)+(1), we use the P-martingale property of M. □

[Application] (Construction of a weak solution of SDE)

Let $b(t, x)$ be a bounded Borel measurable function on $[0, 1] \times \mathbb{R}$ and consider 1-dimensional SDE:

$$dX_t = dB_t + b(t, X_t)\, dt, \tag{1}$$

where B_t is a Brownian motion under P. Then, set under P

$$M_t := \exp\left\{ \int_0^t b(s, B_s)\, dB_s - \tfrac{1}{2} \int_0^t b(s, B_s)^2\, ds \right\}$$
$$dQ := M_1 dP.$$

Then, by taking $\eta_t = \int_0^t b(s, B_s)\, ds$ in Theorem 18.1, we see that

$$\hat{B}_t := B_t - \int_0^t b(s, B_s)\, ds \tag{2}$$

is a Brownian motion under Q.

However, (2) is rewritten as

$$dB_t = d\hat{B}_t + b(t, B_t)\,dt, \quad B_0 = 0.$$

In other words, B_t is a weak solution of the SDE (1) on a probability space $(\Omega, \mathcal{F}, Q)$ equipped with the Brownian motion $\hat{B}$. The method used here is called the transform of drift.

This method is useful also to show the uniqueness of the weak solution. i.e. for $i = 1, 2$, if $(\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)})$, $(X^{(i)}, B^{(i)})$ are both solutions of the SDE (1), then the distributions of $X^{(1)}$ and $X^{(2)}$ on a path space $W^1 \equiv C([0, \infty), \mathbb{R})$ coincide. (one can take $[0, T]$ as the time interval instead of $[0, 1]$ for ${}^\forall T > 0$.) ($\rightarrow$ by similar transformation with $-b$ instead of b, one can reduce to Brownian motion. The details are omitted.)

§19 Ornstein-Uhlenbeck process — Example of SDE

- Consider SDE for $X_t \in \mathbb{R}^d$ with constant diffusion coefficient and linear drift term:

$$dX_t = \alpha dB_t + AX_t dt, \tag{3}$$

  where α and A are $d \times d$ real matrices, and $B = (B_t)$ is a d-dimensional Brownian motion.
- In physics, (3) is a kind of (simplest) Langevin equation.
- The SDE (3) can be solves in terms of the exponential function e^{tA} of matrices as follows:

$$X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A}\alpha dB_s. \tag{4}$$

  This is called Duhamel's formula.

[Proof] One can check that X_t given by (4) is a solution of (3) by applying Itô's formula and noting $e^{0A} = I$ as follows:

$$dX_t \underset{\text{Itô}}{=} Ae^{tA}X_0 dt + e^{0A}\alpha dB_t + \Big(\int_0^t Ae^{(t-s)A}\alpha dB_s\Big)dt$$
$$= AX_t dt + \alpha dB_t.$$

The uniqueness of (strong) solution is easy, since the coefficients (constant and linear function) are Lipschitz continuous. □

• In particular, assuming that the initial value $X_0 = x \in \mathbb{R}^d$ is given, $X = (X_t)$ is a Gaussian process and for each $t \geq 0$,

$$X_t \overset{\text{law}}{=} N(m_t, V_t) \tag{5}$$

with $m_t = e^{tA}x \in \mathbb{R}^d$ and $V_t = \int_0^t e^{sA}\alpha\alpha^* e^{sA^*} ds \in \mathbb{R}^{d\times d}$, where α^*, A^* denote the transposed matrices of α, A.

[Proof] Since $E[(\text{stochastic integral})] = 0$ in (4), we easily see $E[X_t] = m_t$. To compute the covariance matrix V_t, take $^\forall \xi \in \mathbb{R}^d$ and see that

$$\begin{aligned}
(V_t\xi, \xi) &= E\Big[\Big(\int_0^t e^{(t-s)A}\alpha dB_s, \xi\Big)^2\Big] \\
&= E\Big[\Big\{\int_0^t (dB_s, \alpha^* e^{(t-s)A^*}\xi)\Big\}^2\Big] \\
&\underset{\text{Itô isometry}}{=} \int_0^t \Big|\alpha^* e^{(t-s)A^*}\xi\Big|^2 ds \quad (|\cdot| = \text{Euclidean norm}) \\
&= \int_0^t (\alpha^* e^{sA^*}\xi, \alpha^* e^{sA^*}\xi) ds = \int_0^t (e^{sA}\alpha\alpha^* e^{sA^*}\xi, \xi) ds. \qquad \square
\end{aligned}$$

- In particular from (5), if V_t is positive definite (e.g. if $\alpha = 0$, then $V_t = 0$ so that degenerate), we have

$$P_x(X_t \in dy) = \frac{1}{(2\pi)^{d/2}(\det V_t)^{1/2}} e^{-\frac{1}{2}(x-m_t, V_t^{-1}(x-m_t))} dy.$$

  This is called Mehler's formula.

- Let us study the limit distribution of X_t as $t \to \infty$. For simplicity, assume two conditions:
  - $\cdot$ $[\alpha, A] = 0$, i.e. $\alpha A - A\alpha = 0$, α and A are commutative.
  - $\cdot$ A is a symmetric matrix and $A < 0$ (i.e. negative definite).

  Then, the distribution of X_t converges weakly as $t \to \infty$ to $N(0, \alpha\alpha^*(-2A)^{-1})$ which is independent of the starting point x of X_t.

☺ By $A < 0$, we have $m_t \to 0$. For the covariance, noting $[\alpha, A] = [\alpha^*, A^*] = [\alpha^*, A] = 0$, we have

$$V_t = \int_0^t \alpha\alpha^* e^{2sA} ds$$
$$= \alpha\alpha^* (2A)^{-1}(e^{2tA} - I) \to \alpha\alpha^*(-2A)^{-1}. \qquad \square$$

- In fact, in general, instead of the above two conditions, if real parts of all eigenvalues of A are negative, noting $\|e^{sA}\| \le Ce^{-cs}$, $c, C > 0$, one can show that

  $$V_t \to V_\infty := \int_0^\infty e^{sA}\alpha\alpha^* e^{sA^*} ds,$$

  see Karatzas-Shreve p.357, (6.19), or Lunardi Trans. AMS, 1997.

[Definition 19.1] We call the solution $X = (X_t)_{t \geq 0}$ of SDE (or Markov process in general) ergodic or positive recurrent, if the distribution of X_t converges weakly as $t \to \infty$ to a certain probability distribution independently of the initial distribution (distribution of X_0). □

[Remark] Original definition of ergodicity is that the sample mean (time average) converges to the ensemble mean (average under a certain probability measure μ), that is, for ${}^\forall \varphi \in C_b(\mathbb{R}^d)$,

$$\frac{1}{T} \int_0^T \varphi(X_t) dt \underset{T \to \infty}{\longrightarrow} \int_{\mathbb{R}^d} \varphi(x) \mu(dx), \quad a.s. \omega.$$

Probabilistically, this corresponds to the strong law of large numbers (LLN). □

[Remark] The distribution of the Brownian motion

$$\frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} dx$$

spreads and, differently from the Ornstein-Uhlenbeck process, does not converge to a probability measure as $t \to \infty$.

Brownian motion is

· null recurrent when $d = 1, 2$

· transient (not recurrent) when $d \geq 3$.

"recurrent" means that X_t hits any open set (a.s.).
"null" means that the limit distribution is zero measure.