Lecture 19. • Seiberg-Witten invariants of symplectic manifolds

• Definition of Khovanov homology → • exotic $\mathbb{R}^4$

• Milnor's conjecture

....

• SPC4

(X, ω) : symplectic 4-manifold

$J : T_*X \to T_*X$: almost complex structure. $(J^2 = -1)$

We say J is <u>ω-tame</u> if $\omega(v, Jv) > 0$ $\forall v \neq 0$

J is <u>ω-compatible</u> if $\omega(v, Jv) > 0$ $\forall v \neq 0$,

$\omega(-, J-)$ $\qquad \omega(v, Ju) = \omega(u, Jv)$

(Conjecture (Donaldson)) If J is ω-tame, then J is ω'-compatible for some symplectic form ω'. (Tame v.s. compatible)

Conjecture is wide open. (Known when $M = \mathbb{CP}^2$ (Gromov, Taubes))

$b^+(M) = 1$ J generic. (Taubes).

Fact: The space $S_\omega = \{\omega\text{-compatible } J\}$ is contractible.

We pick $J \in S_\omega$. Set $g(u,v) = \omega(u, Jv)$.

Given J, we have a canonical spin^c structure $\mathfrak{S}_J$

$$T^*X \otimes \mathbb{C} = T^{0,1}X \oplus T^{1,0}X$$

$$\Lambda^m T^*X \otimes \mathbb{C} = \bigoplus_{p+q=m} T^{p,q}X \qquad \Gamma(T^{p,q}X)$$

locally $f dz_{i_1} \wedge \cdots dz_{i_p} \wedge d\bar{z}_{i_{p+1}} \wedge \cdots d\bar{z}_{i_{p+q}}$

The spinor bundle $S = \bigoplus_{0 \le q \le 2} T^{0,q}X$

$S^+ = T^{0,0}X \oplus T^{0,2}X = \underline{\mathbb{C}} \oplus K^{-1}$ $\quad S^- = T^{0,1}X \quad (\cong (T_*X, J))$

$c_1(\mathfrak{s}_J) = c_1(S^\pm) = c_1(T_*X)$

$K := T^{2,0}X$

$\rho(v)^2 = -|v|^2 \quad \rho(v)\rho(w) = -\rho(w)\rho(v)$ if $v \perp w$

Clifford multiplication $\rho : T^*X \to \mathrm{End}_{\mathbb{C}}(S)$

$\rho(\alpha)\cdot\beta = \sqrt{2}(\underbrace{\alpha^{0,1}\wedge\beta}_{\Omega^{0,p+1}} - \underbrace{\alpha^{0,1} \lrcorner \beta}_{\Omega^{0,p-1}})$ $\quad \beta \in \Omega^{0,p}$

Here $\alpha = \underbrace{\alpha^{0,1}}_{\Omega^{0,1}} + \underbrace{\alpha^{1,0}}_{\Omega^{1,0}}$

$\alpha^{0,1} \lrcorner (\underbrace{\beta_1}_{\text{1-forms}} \wedge \beta_2 \wedge \cdots \wedge \beta_k) := \sum_{1\le i\le k} (-1)^{i-1} \beta_1 \wedge \cdots \wedge \langle \beta_i, \alpha^{0,1}\rangle \wedge \cdots \wedge \beta_k$

(This works for any J and any dimension.)

Theorem (Taubes) Given (X,ω) with $b^+(X) > 1$, we have

- $SW(\mathfrak{s}_J) = \pm 1$, $SW(\bar{\mathfrak{s}}_J) = \pm 1$. (Recall $c_1(\mathfrak{s}_J) = c_1(T_*X)$)
- Given any $\alpha \in H^2(X;\mathbb{Z})$,

  then $SW(\mathfrak{s}_J + \alpha) \neq 0$ only if $0 \le \omega\cdot\alpha \le \omega \cdot c_1(K)$

Here "=" is achieved iff $\alpha = 0$ or $\alpha = c_1(K)$. $\quad c_1(K) = -c_1(T_*X)$

$S^\pm_{\mathfrak{s}_J+\alpha} = S^\pm_{\mathfrak{s}_J} \otimes E \quad c_1(E) = \alpha$

This gives very strong constraint on the Seiberg-Witten invariant of symplectic 4-mfd.

Pick $p \in X$, pick local chart $U \to \mathbb{R}^n$, s.t. the following $\boxed{\text{holds at } p}$

- $\{\frac{\partial}{\partial x_i}(p)\}$ is an orthonormal basis of T_pX
- $(\nabla^{LC}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j})(p) = 0 \quad \forall i,j.$

$$z_1 = x_1 + ix_2 \qquad z_2 = x_3 + ix_4$$

- Set $e_i = \frac{\partial}{\partial x_i}(p)$, $e^i = dx_i(p)$. Then $J: e_1 \to e_2 \quad e_2 \to -e_1$

$$e_3 \to e_4 \quad e_4 \to -e_3$$

$$T_p^{0,0} \oplus T_p^{0,2}$$
$$\|$$

Pick orthonormal basis of $S_p^+ = \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot\left(-\frac{d\bar{z}_1 \wedge d\bar{z}_2}{2}\right)$

$$T_p^{0,1} = S_p^- = \mathbb{C}\cdot\frac{d\bar{z}_1}{\sqrt{2}} \oplus \mathbb{C}\frac{d\bar{z}_2}{\sqrt{2}}$$

Then we can check $\{\rho(e_i)\}$ is "standard" i.e.

$$\rho(e_1) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad \rho(e_2) = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \quad \rho(e_2) = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \rho(e_3) = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Using this, we can further check

$$dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$
$$\|$$
$$\rho(e^1 \wedge e^2 + e^3 \wedge e^4) = \begin{pmatrix} -2\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \rho(\omega) = -2\sigma_1$$
$$\|$$
$$\omega(p)$$

So given $\phi = (\alpha, \beta) \in \Gamma(S^+) = \Gamma(\overset{\overset{1}{\uparrow}}{\underline{\mathbb{C}}}) \oplus \Gamma(\overset{\overset{0}{\uparrow}}{K^{-1}})$

we have $\rho(\omega)(\alpha, \beta) = (-2i\alpha, 2i\beta)$

$\boxed{\mathrm{End}(S^+)}$
$\quad\Psi$

Note $(\phi\,\phi^*)_0 = \begin{pmatrix} \frac{|\alpha|^2-|\beta|^2}{2} & \alpha\bar\beta \\ \bar\alpha\beta & \frac{|\beta|^2-|\alpha|^2}{2} \end{pmatrix}$

Set $\phi_0 = (1, 0)$ Then

$$\boxed{\rho^{-1}(\phi_0^*\phi_0)_0 = \rho^{-1}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{-i\omega}{4}}$$

Claim: $\exists$ a unique spin^c connection A_0
s.t. $\nabla_{A_0}\phi_0 \in \Omega^1(X, K^{-1})$ (i.e. $\langle \nabla_{A_0}\phi_0, \phi_0\rangle = 0$)

Actually, given any A, we have $\nabla_A\phi_0 \in \Omega^1(X, \underline{\mathbb{C}}\oplus K^{-1})$

$$0 = d|\phi_0|^2 = \langle \nabla_A\phi_0, \phi_0\rangle + \langle \phi_0, \nabla_A\phi_0\rangle$$
$$= 2\mathrm{Re}\langle \nabla_A\phi_0, \phi_0\rangle$$

so $\langle \nabla_A\phi_0, \phi_0\rangle \in \Omega^1(X; i\mathbb{R})$

Set $A_0 = A - \nabla_A\phi_0$. Then $\langle \nabla_{A_0}\phi_0, \phi_0\rangle = 0$.

(If (X, ω) is Kahler, then $\nabla_{A_0}\phi_0 = 0$)

Claim: $D_{A_0}\phi_0 = 0$ This is done via calculation.

Now we consider the Seiberg-Witten equation

$$\begin{cases} F_A^+ = \rho^{-1}(\phi^*\phi)_0 + i\eta \\ D_A\phi = 0 \end{cases}$$

$\eta \in \Omega^2_+(X)$ perturbation form

Recall $\rho^{-1}(\phi_0^*\phi_0) = \frac{-i\omega}{4}$ $(A_0, \sqrt{r}\phi_0)$ is solution

So if we choose $\eta = -iF^+_{A_0} - \underline{\underline{r}}\omega \qquad r \in \mathbb{R}^{\geq 0}$

Then $(A_0, \sqrt{r}\phi_0)$ will be a solution.

Taubes actually proved that if $r \gg 0$, then $(A_0, \sqrt{r}\phi_0)$ is the unique solution. So $SW(\mathfrak{s}_J) = \pm 1$.
By conjugation symmetry $SW(\bar{\mathfrak{s}}_J) = \pm 1$.

Now consider $\mathfrak{s} = \mathfrak{s}_J + \alpha$. Then $S^+ = \overset{(\mathbb{C} \oplus K^{-1}) \otimes E}{E \oplus K^{-1}E}$ with $c_1(E) = \alpha$.
use the same perturbation $\eta = -iF^+_{A_0} - r\omega$

Taubes: Given $r_i \to +\infty$, solutions $(A_i, (\alpha_i, \beta_i))$

$\alpha_i \in \Gamma(E) \qquad \beta_i \in \Gamma(K^{-1}E)$

Taubes proved that as $i \to +\infty$

$\alpha_i^{-1}(0) \longrightarrow$ J-holomorphic curve $\quad f: C \longrightarrow X$

where C is a Riemanian surface and $df \circ J = J \circ df$

$\mathfrak{s}_J + \alpha = \mathfrak{s}_J$

In particular, $[\omega][f(C)] = \int_C \omega = \text{volume of } C \geq 0$

However, $[\underline{\underline{f_*}}[C]] = \text{P.D.} E = \text{P.D.} \alpha$

So $\omega \cdot \alpha \geq 0$ and $\omega \cdot \alpha = 0 \Leftrightarrow C$ is empty $\Leftrightarrow \alpha = 0$.

I.e. if $\omega \cdot \alpha < 0$, then for $r \gg 0$, SW has no solution.

Note

$$SW(\mathfrak{s}_J+\alpha) \overset{\pm}{=} SW(\overline{\mathfrak{s}_J+\alpha}) = SW(\mathfrak{s}_J+(c_1(K)-\alpha))$$

so $SW(\mathfrak{s}_J+\alpha)\neq 0 \Rightarrow \omega\cdot(c_1(K)-\alpha)\geq 0$

"=" is achieved iff $c_1(K)-\alpha=0$.

$$0 \leq \omega\cdot\alpha \leq \omega\cdot c_1(K)$$

Using this argument (and very delicate analysis) Taubes proved:

$$SW(\mathfrak{s}_J+\alpha) = Gr(\alpha)$$

$$D_{A_0+a}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0 \iff \bar{\partial}_a\alpha+\bar{\partial}_a^*\beta=0$$

Gromove invariant that counts

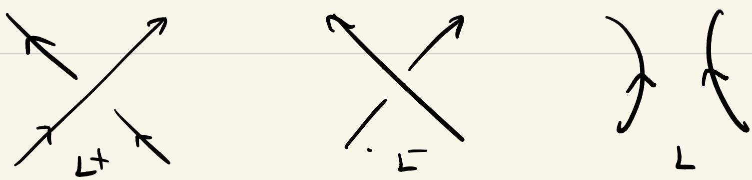
$\#\{$psedo holomorphic $C\to X \mid [C]=\alpha\}$

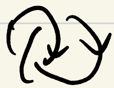
genus ≥ 0, not necessarily connected.

Jones polynomial

Given oriented link L, the Jones polynomial $J_L(t)$ is defined by the skein relation • $J_O(t)=1$

• $t^{-1}J_{L_+}(t) - tJ_{L_-}(t) = (t^{\frac{1}{2}}-t^{-\frac{1}{2}})J_{L_0}(t)$



E.g. J  $(t) = t + t^3 - t^4$.

$\leftarrow 3_1$

We define a variation $\tilde{J}_L(q) = (q+q^{-1})\, J_L(q^2)$

so $\tilde{J}_{3_1}(q) = (q+q^{-1})(q^2+q^6-q^8) = q+q^3+q^5-q^9$

Khovanov homology $Kh(L) = \bigoplus_{i,j\in\mathbb{Z}} Kh^{i,j}(L)$ is a "categoryfication" of $\tilde{J}(t)$. That means

i ← homological grading

j ← quantum / Jones grading

- $\chi(Kh^{\bullet,\bullet}(L)) := \sum_{i,j} (-1)^i q^j \operatorname{rank} Kh^{i,j}(L) = \tilde{J}_L(q)$

- Kh is a functor $Kh : Cob \to Ab/\pm$

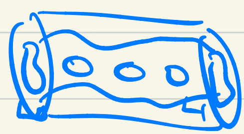
Here $Ob(Ab/\pm) = \{\text{abelian groups}\}$

$Mor_{Ab/\pm}(G_1, G_2) = \{\text{homomorphisms } G_1 \to G_2\}/\pm$

object $cob = \{\text{oriented links in } S^3\}$

$\{\text{Morphisms from } L_0 \text{ to } L_1\} = \{\text{cobordism } \Sigma \text{ from } L_0 \text{ to } L_1\}$

I.e. Σ (← oriented.) $\hookrightarrow [0,1]\times S^3$, $\partial F = \{0\}\times -L_0 \sqcup \{1\}\times L_1$



So a cobordism Σ from L_0 to L_1 induces a map

$F_\Sigma : Kh^{i,j}(L_0) \to Kh^{i,j+\chi(\Sigma)}(L_1)$, which is well-defined up to a sign. (Khovanov, Jacobsson)

Now we define $Kh(L)$.

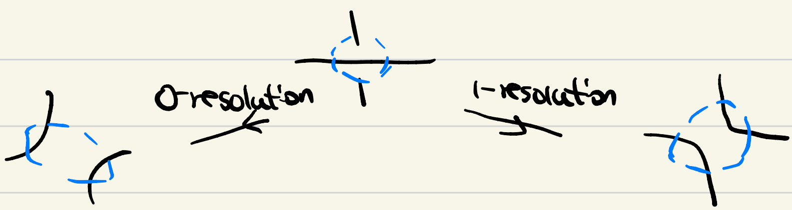
Convention: Given a bi-graded abelian group / chain complex $C = \{C^{i,j}\}_{i,j\in\mathbb{Z}}$, we define $C[s]\{t\}$ to be the shifting of C "up" by degree (s,t).

I.e. $C[s]\{t\}^{i,j} := C^{i-s,\,j-t}$.

(In our setting i = homological grading, j = quantum grading)

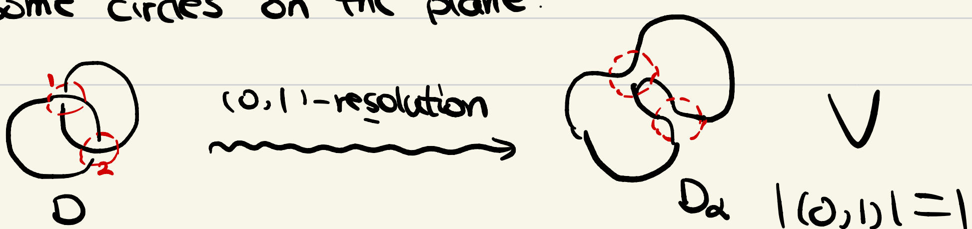
- L: oriented link
- D: oriented link diagram
- $n_+ :=$ # positive crossings in D

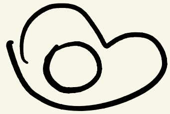
  $n_- :=$ # negative crossings in D $\qquad n = n_+ + n_-$
- We order the crossings of D by $1, 2, \cdots, n$
- For any crossing $\in \{1, \cdots, n\}$, there are two resolutions



(This doesn't depend on orientation of L)

So for any $\alpha \in \{0,1\}^n$, we have a complete resolution of D into some circles on the plane.





$V \otimes V$

We consider the cube $[0,1]^n$. Then

- To each vertex $\alpha \in \{0,1\}^n$, we assign an abelian group.

$V_\alpha(D) := \bigotimes_{c(\alpha)} V\{|\alpha|\}$ $\qquad |\alpha| = \#1\text{s in } \alpha$

$V = \mathbb{Z}\langle v_+, v_- \rangle$, $\qquad$ quantum grading of $v_\pm = \pm 1$

homological grading of $v_\pm = 0$

$c(\alpha) = \{\text{circles in } \alpha\}$

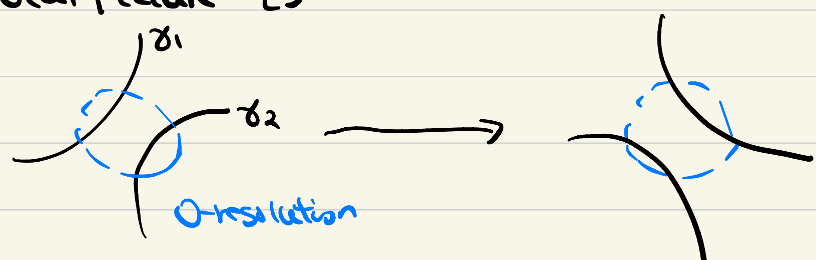
- To each edge $\xi : \alpha \to \alpha'$, we assign a linear map

$d_\xi : V_\alpha \to V_{\alpha'}$ as follows:

$\alpha = (\alpha_0, -- \alpha_i, 0, \alpha_{i+1}, ---, \alpha_n)$

$\alpha' = (\alpha_0, --- \alpha_i, 1, \alpha_{i+1}, ---, \alpha_n)$

so local picture is



Two possibilities, 1) γ_1, γ_2 belongs to 2 circles. Then ξ is a merge of 2 circles c_1, c_2 in D_α to a single circle c in $D_{\alpha'}$

We define $m : V_{c_1} \otimes V_{c_2} \to V_c$ by

$$m : \begin{cases} v_+ \otimes v_+ \to v_+ & v_- \otimes v_+ \to v_- \\ v_+ \otimes v_- \to v_- & v_- \otimes v_- \to 0 \end{cases}$$

$\mathbb{Z}[u]/u^2$

$v_- = u \quad v_+ = 1$

$V_\alpha(D) \to V_{\alpha'}(D)$

Set $d_\xi = \left(\bigotimes_{C(\alpha)-\{C_1,C_2\}} \mathrm{Id}\, V \right) \otimes m$

$m: V \otimes V \to V.$

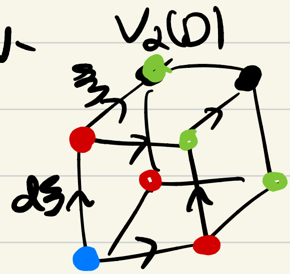
2) γ_1, γ_2 belongs to the same circle c in D_α

Then ξ splits c into 2 circles c_1, c_2 in $D_{\alpha'}$.

We define the splitting map $m: V_c \to V_{c_1} \otimes V_{c_2}$.

$$m: \begin{cases} V_+ \mapsto V_- \otimes V_+ + V_+ \otimes V_- \\ V_- \mapsto V_- \otimes V_- \end{cases}$$

Set $d_\xi = \bigotimes_{C(\alpha)-\{c\}} \mathrm{Id}\, V \otimes m.$



- We form a chain complex $(|D|, d)$, where

$$|D|^r = \bigoplus_{|\alpha|=r} V_\alpha(D)$$

$d: |D|^r \to |D^{r+1}|$ $\quad d = \sum_{\substack{\xi:\alpha_0 \to \alpha_1 \\ |\alpha_0|=r}} (-1)^{|\xi|} d_\xi$

Here if $\xi: (\alpha_1, \dots \alpha_i, 0, \alpha_{i+1}, \dots \alpha_n) \to (\alpha_1, \dots, \alpha_i, 1, \alpha_{i+1}, \dots \alpha_n)$

Then $|\xi| = \sum_{l \le i} \alpha_l$

Thm: (1) $d^2 = 0$

(2) The homology of $\{|D|[-n_-]\{n_+ - 2n_-\}, d\}$ is a link invariant, we define it to be $Kh^{\cdot,\cdot}(L)$.