YMSC Lectures Week 2

September 2024

1 Interactions with superpolynomial decay

1.1 Gauge-fixing interactions

First, we need to standardize the choice of regions X on which $\Phi(X)$ depends so that the derivation defined by an interaction Φ uniquely determines the observables $\Phi(X) \in \mathscr{A}_X$. We will refer to this as "gauge-fixing", because the freedom to choose $\Phi(X)$ for a fixed derivation is somewhat analogous to the freedom to choose a vector potential for a given magnetic field.

Definition 1. A brick is a subset of $\mathbb{Z}^d$ of the form $\{(x_1, \ldots, x_d) : n_i - 1/2 \leq x_i < m_i - 1/2,\ i = 1, \ldots, d\}$, where n_i and m_i are integers satisfying $n_i < m_i$. The empty brick is the empty subset.

We denote the set of all bricks in $\mathbb{Z}^d$ together with the empty brick by $\mathbb{B}_d$. $\mathbb{B}_d$ is a poset (partially ordered set) with respect to inclusion. That is, the relation of inclusion for bricks is reflexive ($X \subseteq X$), transitive ($X \subseteq Y \subseteq Z$) and anti-symmetric (if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$).

Next, we eliminate the ambiguity in how we assign "support" to a local observable. The issue is that 1 is shared by all $\mathscr{A}_j$, so one cannot tell where it is localized. We eliminate this ambiguity by requiring all $\Phi(X)$ to be traceless. We can do this by subtracting from each $\Phi(X)$ a multiple of its trace which does not affect the derivation. So from now on all $\Phi(X)$ are traceless and anti-self-adjoint. Also, it has a well-defined support (which may be smaller than X) which can be visualized using a "Pauli basis" for $\mathscr{A}_\ell$. This is a basis obtained by choosing an orthonormal self-adjoint basis $\mathcal{E}_j^k$, $k = 0, \ldots, d_j^2 - 1$ for each $\mathscr{A}_j$ (with respect to the usual inner product $(a, b) = \mathrm{Tr}(a^* b)$) so that $\mathcal{E}_j^0$ is the identity element in $\mathscr{A}_j$. The resulting basis elements of $\mathscr{A}_\ell$ can be labeled by functions $\nu : \Lambda \to \mathbb{N}_0$ with $\nu(j) < d_j^2$ which vanish outside of a finite set. The identity element in $\mathscr{A}_\ell$ corresponds to ν being identically zero. If we denote by $\mathrm{supp}(\nu) \in \mathrm{Fin}(\Lambda)$ the support of ν, then the support of any traceless local observable is the union of supports of its components in the Pauli basis.

Given any interaction Φ, we can define a physically equivalent interaction Φ' by declaring that $\Phi(X)$ is nonzero only when X is a brick, and for $X \in \mathbb{B}_d$

letting

$$\Phi'(X) = \sum_{Y\subseteq X}{}' \Phi(Y),$$

where the prime means that Y cannot be a subset of any brick is which is a proper subset of X. Every finite subset A of $\mathbb{Z}^d$ is a subset of some brick, so this new interaction Φ' gives the same derivation as Φ and thus is physically equivalent.

Now we can finally define an interaction as a function from $\mathbb{B}_d$ to $\mathscr{A}_\ell$ which is uniquely determined by the corresponding derivation. Namely, for any brick X $\Phi(X)$ should be anti-self-adjoint, traceless, localized on the brick X, and not localized on any brick which is a proper subset of X.

1.2 UAL derivations

To make the derivation well-defined at least on $\mathscr{A}_\ell$, we need to demand that $\Phi(X)$ decay sufficiently rapidly when X is large. One option is to say that $\Phi(X) = 0$ if $\mathrm{diam}(X) > R$, where $R > 0$ is some number. Such derivations correspond to finite-range interactions. But what is R? We should probably allow arbitrary R. But then the space of all such derivations does not form a nice topological vector space.

Instead, we demand superpolynomial decay:

Definition 2. An interaction $\Phi : \mathbb{B}_d \to \mathscr{A}_{a\ell}$ satisfying the above conditions is a UAL derivation (uniformly almost local derivation) if

$$\sup_{X\in\mathbb{B}_d} \|\Phi(X)\|(1 + \mathrm{diam}(X))^\alpha = C_\alpha < \infty$$

for any non-negative integer α.

The above conditions define a nice topological vector space: a Fréchet space.

1.3 Fréchet spaces

A seminorm on a real or complex vector space V is a map $V \to \mathbb{R}$, $v \mapsto \|v\|$ such that $\|v\| \geq 0$ for all $v \in V$, $\|v + v'\| \leq \|v\| + \|v'\|$ for all $v, v' \in V$, and $\|cv\| = |c|\|v\|$ for all $v \in V$ and all scalars c. A seminorm is a norm if $\|v\| = 0$ implies $v = 0$.

A Fréchet space is a complete Hausdorff topological vector space whose topology is determined by a countable family of seminorms $\|\cdot\|_\alpha$, $\alpha \in \mathbb{N}_0$. A base of neighborhoods of zero for such a topology consists of sets

$$U_{(\alpha_1,\varepsilon_1)\dots(\alpha_n,\varepsilon_n)} = \{v \in V : \|v\|_{\alpha_i} < \varepsilon_i,\ i = 1,\dots,n\}, \tag{1}$$

where $n \in \mathbb{N}$, $\alpha_i \in \mathbb{N}_0$, and $\varepsilon_i > 0$. Any finite-dimensional Euclidean vector space is a special case where all the seminorms happen to be the same and equal to the Euclidean norm.

We will be often dealing with a situation where the seminorms satisfy $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$. One calls such Fréchet spaces graded Fréchet spaces. Then the sets $U_{\alpha,\varepsilon} = \{v \in V : \|v\|_\alpha < \varepsilon\}$, $\alpha \in \mathbb{N}_0$, also form a base of neighborhoods of zero. A linear map $f : V \to V'$ between graded Fréchet spaces is continuous iff for any $\alpha \in \mathbb{N}_0$ there is a $\beta \in \mathbb{N}_0$ and a constant C_α such that $\|f(v)\|_\alpha \leq C_\alpha \|v\|_\beta$. The Cartesian product of two graded Fréchet spaces V, V' is also a graded Fréchet space, with the seminorms $\|(v, v')\|_\alpha = \|v\|_\alpha + \|v'\|'_\alpha$.

Different families of seminorms on V may define the same topology; in that case one says that the families are equivalent. A family of seminorms $\|\cdot\|'_\beta$, $\beta \in \mathbb{N}_0$ is equivalent to a family $\|\cdot\|_\alpha$, $\alpha \in \mathbb{N}_0$, if for any β there is an α and a constant C_β such that $\|\cdot\|'_\beta \leq C_\beta \|\cdot\|_\alpha$, and vice versa, for any α there is an β and a constant C'_α such that $\|\cdot\|_\alpha \leq C'_\alpha \|\cdot\|'_\beta$.

If X is a compact topological space and V is a graded Fréchet space, then the space $C(X, V)$ of continuous V-valued functions on X is also a graded Fréchet space. The corresponding family of seminorms is $\|f\|_\alpha = \sup_{x \in X} \|f(x)\|_\alpha$, $\alpha \in \mathbb{N}_0$. In the case when $X = [a, b] \subset \mathbb{R}$ elements of $C([a, b], V)$ are called continuous curves in V. Most basic rules of calculus (such as the existence of integrals of continuous functions, the Fundamental Theorem of Calculus, the Mean Value Theorem, the continuous dependence of integrals of continuous functions on parameters, etc.) hold in the setting of continuous functions on regions in $\mathbb{R}^n$ valued in a Fréchet space V.

Going back to our problem, the space of UAL derivations becomes a Fréchet space if we say that

$$\|\Phi\|_\alpha = \sup_{X \in \mathbb{B}_d} \|\Phi(X)\|(1 + \operatorname{diam}(X))^\alpha, \quad \alpha = 0, 1, 2, \ldots$$

In this case all seminorms are actually norms. It is also clear that $\|\Phi\|_\alpha \leq \|\Phi\|_\beta$ if $\alpha < \beta$, so this is a graded Fréchet space.

2 Almost local observables

UAL derivations are defined everywhere on $\mathscr{A}_l$:

$$\mathsf{F} : a \mapsto \sum_{X \in \mathbb{B}_d} [\Phi(X), a], \quad a \in \mathscr{A}_Y.$$

The sum over X is rapidly convergent because

$$\begin{aligned}\sum_{\operatorname{diam}(X)=R, X \cap Y \neq 0} \|[\Phi(X), a]\| \leq 2\|a\||Y| \sup_j \sum_{\operatorname{diam}(X)=R, X \ni j} \|\Phi(X)\| \\ \leq 2\|a\||Y| C_\alpha \frac{N(R)}{(1+R)^\alpha}, \quad (2)\end{aligned}$$

where $N(R)$ is the number of bricks of diameter R containing site j (which grows as some power of R) and α is an arbitrary integer. Thus the remainder in the sum over bricks of diameter $R \geq r$ is $\mathcal{O}(r^{-\infty})$.

Moreover, it is fairly clear that $\mathsf{F}(a)$ is itself well localized, i.e. it can be approximated by a local observable supported on a ball of size r with an $\mathcal{O}(r^{-\infty})$ error. We will call such observables almost local. Here is a precise definition.

First, recall that we gave a tracial state on $\mathscr{A}$. Instead of tracing over all sites, we can trace over sites outside any region Y. This gives a "conditional expectation value" (conditional on whatever happens in Y) which is a positive map $\mathscr{A} \to \mathscr{A}_Y$. We will only use it for finite Y, so that $\mathscr{A}_Y$ is a matrix algebra. Let $a|_Y \in \mathscr{A}_Y$ denote the conditional expectation value of a. The same arguments as for states show that such maps cannot increase the norm, $\|a|_Y\| \leq \|a\|$.

Definition 3. An observable $a \in \mathscr{A}$ is called almost local if for some $j \in \mathbb{Z}^d$ and all $\alpha \in \mathbb{N}$

$$\|a\|^{cev}_{j,\alpha} := \sup_r (1+r)^\alpha \|a - a|_{B_j(r)}\| < \infty. \tag{3}$$

Here $B_j(r)$ is a ball of radius r centered at j.

Any such observable can be written as a sum over local observables localized on $B_j(r)$, $r = 1, 2, \ldots$ whose norm is $\mathcal{O}(r^{-\infty})$. This is exactly the result of application of F to a local observable.

Let us denote by $\mathscr{A}_{a\ell}$ the space of almost local observables. It is itself a Fréchet space. There are many other equivalent ways to define the same topology using "equivalent" norms. For example, all choices of j give equivalent norms. Or instead of using $a|_j$ one can use the best approximation of a by a local observable on $B_j(r)$, i.e. define

$$\|a\|_{j,\alpha} := \sup_r (1+r)^\alpha \inf_{b \in \mathscr{A}_{B_j(r)}} \|a - b\|.$$

It is clear that $\|a\|_{j,\alpha} \leq \|a\|^{cev}_{j,\alpha}$. One can also show that

$$\|a\|^{cev}_{j,\alpha} \leq C'_\alpha \|a\|_{j,\alpha+2d+1}$$

for some constants C'_α.

Now it is straightforward to see that for any $a \in \mathscr{A}_{a\ell}$ $\mathsf{F}(a)$ is also almost local. This is left as an exercise.

3 Density of a UAL derivation

A finite-range interaction Φ has a "density":

$$\Phi_j = \sum_{X \ni j} \frac{1}{|X|} \Phi(X).$$

This is a local observable localized on some region near j. More precisely, if Φ has range R, then $\Phi_j \in B_j(R)$. Then the corresponding derivation is written in a more "physics-friendly" form:

$$\delta^\Phi(a) = \sum_j [\Phi_j, a].$$

We can do the same thing with UAL derivations. The only difference is that the density of F at site j will now be an infinite sum, so the resulting observable F_j will be only almost local. I will omit a detailed proof, since this is rather clear. So, a UAL derivation can be thought of as a formal infinite sum

$$\sum_{j\in\mathbb{Z}^d} \Phi_j$$

where each Φ_j is an almost local traceless anti-self-adjoint observable.

Turns out there is a converse result: every such formal sum (where every F_j is an almost local observable "confined" at j to the same accuracy for all j) is a UAL derivation. The proof is somewhat nontrivial. First one decomposes every almost local observable into a sum over local observables localized on bricks:

$$a = \sum_{X\in\mathbb{B}_d} a^X.$$

Since a is almost local, one can show that a^X decays superpolynomially with $\mathrm{diam}(j\cup X)$ (this is the nontrivial bit). One can do it for every F_j. Then one sums the contribution from every j:

$$\mathsf{F}^Y = \sum_j \mathsf{F}_j^Y, \quad Y\in\mathbb{B}_d.$$

This sum is rapidly convergent and it is easy to see that it decays superpolynomially with $\mathrm{diam}\, Y$.

One disadvantage of thinking about a UAL derivation in terms of densities is that the map from densities to UAL derivations is many-to-one: if one picks two sites k, k' and adds an almost local observable to F_k and subtracts the same observable from $\mathsf{F}_{k'}$, the sum and the derivation are unchanged.

4 Densities and currents

In our approach, generators of symmetries are UAL derivations. They can be thought of as formal sums

$$\mathsf{F} = \sum_j \mathsf{F}_j$$

where $\mathsf{F}_j \in \mathscr{A}_{a\ell}$ is an almost local observable approximately localized at site $j\in\mathbb{Z}^d$. The corresponding derivation is

$$\mathsf{F}(a) = \sum_j [\mathsf{F}_j, a]$$

In our conventions, we require F_j to be anti-self-adjoint, $\mathsf{F}_j^* = -\mathsf{F}_j$. Also, since adding multiples of identity to F_j does not affect the derivation F, it s convenient to normalize F_j so that it is traceless.

We will refer to the collection of observables F_j, $j \in \mathbb{Z}^d$, as a density of a UAL derivation F. There is an obvious ambiguity in the density: if we replace

$$\mathsf{F}_j \mapsto \mathsf{F}'_j = \mathsf{F}_j + \sum_i \mathsf{G}_{ij}$$

then the corresponding derivation is unchanged, $\mathsf{F} = \mathsf{F}'$. Here $\mathsf{G}_{ij} \in \mathscr{A}_{a\ell}$ is a collection of observables such that

- $\mathsf{G}_{ij} = -\mathsf{G}_{ji}$
- G_{ij} is anti-self-adjoint and traceless
- G_{ij} is approximately localized on both i and j

The latter condition implies that $\|\mathsf{G}_{ij}\| = O(|i-j|^{-\infty})$, so the sum $\sum_i \mathsf{G}_{ij}$ is rapidly convergent. Later we will see that these are the only ambiguities in F_j.

Quantities like G_{ij} also appear when one considers currents of conserved quantities on a lattice. If Q is a generator of a symmetry (for example, it is could be the electric charge), we expect local conservation:

$$\frac{d\mathsf{Q}_j}{dt} := [\mathsf{H}, \mathsf{Q}_j] = -\sum_i \mathsf{J}_{ij},$$

where J_{ij} represents the flow of charge from site j to site i. Physical considerations demand $\mathsf{J}_{ij} = -\mathsf{J}_{ij}$ as well as other requirements listed above. Thus objects like G_{ij} are lattice versions of currents.

The above local conservation equation is a lattice analog of

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot j.$$

where ρ is the density of charge Q, $Q = \int \rho \; d^d x$. The ambiguity in F_j has a continuum counterpart: one can redefine the density ρ as $\rho \mapsto \rho + \nabla \cdot g$ while only changing Q by surface terms.

In the continuum, there are also separate ambiguities in the current j: one can redefine $j \mapsto j + \nabla \times m$ without changing $\nabla \cdot j$ and therefore without affecting the validity of the conservation equation. This has a lattice counterpart too: we can always redefine

$$\mathsf{J}_{ij} \mapsto \mathsf{J}_{ij} + \sum_k \mathsf{M}_{kij}$$

where $\mathsf{M}_{ijk} \in \mathscr{A}_{a\ell}$ is completely anti-symmetric in all indices, anti-self-adjoint, traceless, and approximately localized on i, j, and k. We will call such objects "magnetizations". Obviously, one can go on, but to define the Hall conductance we will only need the above three types of objects.

Let's give some examples of currents. If $\mathsf{F} = \mathsf{H}$, the Hamiltonian, then we get the energy current J^E_{ij}. An obvious solution to the conservation equation is

$$\mathsf{J}^E_{ij} = -[\mathsf{H}_i, \mathsf{H}_j].$$

Of course, this solution is not unique, there are ambiguities as described above.

Now consider the electric charge Q with a density Q_j. In this case the conservation equation is solved by

$$\mathsf{J}^{el}_{ij} = -[\mathsf{H}_i, \mathsf{Q}_j] + [\mathsf{H}_j, \mathsf{Q}_i].$$

Here we assumed that each H_j is separately $U(1)$-invariant, and thus $[\mathsf{H}_j, \mathsf{Q}] = 0$.

5 The Noether chain complex

It is convenient to use index-free notation and denote the spaces of densities, currents and magnetizations by C_1, C_2, C_3, respectively. We also rename the space of UAL derivations to C_0. We also define maps $\partial_n : C_n \to C_{n-1}$ by

$$(\partial_n \mathsf{F})_{j_1 \dots j_{n-1}} = \sum_{j_0} \mathsf{F}_{j_0 j_1 \dots j_{n-1}}.$$

Obviously we have $\partial_n \circ \partial_{n-1} = 0$. This expresses the fact that $\partial_n \mathsf{G}$ is a possible ambiguity in $\mathsf{F} \in C_{n-1}$ with a fixed $\mathsf{H} = \partial_{n-1}\mathsf{F} \in C_{n-2}$.

From the mathematical viewpoint, this means that the sequence of spaces C_n and the maps ∂_n forms a complex (of Frechet spaces). We will call it the Noether complex, since it encodes charges, their densities, and currents.

As mentioned above, the spaces C_q are Fréchet spaces. The seminorms defining the topology are

$$\|\mathsf{a}\|_\alpha := \sup_{a \in \{0,1,\dots,q\}} \sup_{j_0,\dots,j_q \in \Lambda} \|\mathsf{a}_{j_0 \dots j_q}\|_{j_a,\alpha}, \quad \alpha \in \mathbb{N}_0 \tag{4}$$

The differential ∂ is continous in this topology.

This raises a question: what is the homology of the Noether complex? That is, what is $\ker\partial_{n-1}/\mathrm{im}\partial_n$? Turns out the homology is trivial. Physically, this means that we already described the most general ambiguities in currents and densities.

Theorem 1. The homology of the Noether complex is trivial.

Proof. As usual, to prove the vanishing of homology we need to exibit a contracting homotopy. That is, a linear map h of degree $+1$ which satisfies $h \circ \partial + \partial \circ h = 1$. We claim that the following map works:

$$h_0(\mathsf{A})_j := \sum_{Y \in \mathbb{B}_d} \frac{\chi_j(Y)}{|Y \cap \Lambda|} \mathsf{A}^Y, \tag{5}$$

while for $q > 0$ we let

$$h_q(\mathsf{a})_{j_0 \dots j_{q+1}} = \sum_{Y \in \mathbb{B}_d} \sum_{k=0}^{q+1} (-1)^k \frac{\chi_Y(j_k)}{|Y \cap \Lambda|} \mathsf{a}^Y_{j_0 \dots \widehat{j_k} \dots j_{q+1}}. \tag{6}$$

It is not hard to show that the infinite sums over bricks are convergent and the map h is well-defined. $\square$

6 The algebraic structure of the Noether complex

Definition 4. A differential graded Lie algebra is a chain complex of vector spaces $(C_\bullet, \partial)$ equipped with a bilinear operation (bracket) of degree 0 which satisfies three properties

- Graded skew-symmetry: $[v, w] = -(-1)^{|v|\cdot|w|}[w, v]$;
- Graded Jacobi identity:

$$(-1)^{|u|\cdot|w|}[u,[v,w]] + (-1)^{|w|\cdot|v|}[w,[u,v]] + (-1)^{|u|\cdot|v|}[v,[w,u]] = 0;$$

- Graded Leibniz identity: $\partial[v, w] = [\partial v, w] + (-1)^{|v|}[v, \partial w]$.

I claim that the Noether complex has a bracket if we shift the degree by $+1$, so that the complex starts in degree 0 rather than degree -1. To indicate this, I will define $C'_p = C_{p-1}$.

The bracket is defined as follows.

$$[\mathsf{A}, \mathsf{B}]_{j_1 \dots j_{n+m}} = \sum_\sigma \frac{(-1)^{|\sigma|}}{n!m!} [\mathsf{A}_{\sigma(j_1)\dots\sigma(j_n)}, \mathsf{B}_{\sigma(j_{n+1}\dots\sigma(j_{n+m})}].$$

Here the sum is over all permutations of the indices.

For example, if the energy density is $\mathsf{h} \in C'_1$, the energy current is now simply $\frac{1}{2}[\mathsf{h}, \mathsf{h}] \in C'_2$. The electric current corresponding to a charge density $\mathsf{q} \in C'_1$ is $[\mathsf{h}, \mathsf{q}] \in C'_2$.

7 Locally-Generated Automorphisms

As I already mentioned, every $\mathsf{F} \in \mathfrak{D}_{al}$ can be exponentiated to a well-defined strongly continuous one-parameter group of automorphism $\alpha_\mathsf{F}(t)$. By definition, $\alpha_\mathsf{F}(t)$ satisfies

$$\frac{d\alpha_\mathsf{F}(t)(a)}{dt} = \alpha_\mathsf{F}(t)(\mathsf{F}(a)),$$

where $a \in \mathscr{A}_{a\ell}$ is arbitrary.

Lemma 1. For any $X \subset \mathbb{Z}^d$, let $\Pi_X : \mathscr{A} \to \mathscr{A}$ be partial trace over all degrees of freedom on X. This is a projection map from $\mathscr{A}$ to its sub-algebra $\mathscr{A}_{X^c}$. Then for any $a \in \mathscr{A}$ we have an estimate

$$\|a - \Pi_X(a)\| \leq \sup_{b \in \mathscr{A}_X} \frac{\|[a, b]\|}{\|b\|}$$

Proposition 1. For all t and all $a \in \mathscr{A}_{a\ell}$, we have $\alpha_\mathsf{F}(t)(a) \in \mathscr{A}_{a\ell}$

Sketch of a proof: we want to show that for any almost local observable a the quasi-local observable $\alpha_{\mathsf{F}}(t)(a)$ can be approximated by a local observable on a ball of radius r with an error which decays faster than any negative power of r. We will say that quantities like this are negligible. Take a large $r > 0$. If we replace a with its best approximation on a ball of radius $r/2$, this will change $\alpha_{\mathsf{F}}(t)(a)$ only by a negligible quantity. So we might as well take a to be a local observable on a ball of radius $r/2$. The Lieb-Robinson bound tells us that the commutator of $\alpha_{\mathsf{F}}(t)(a)$ with any $b \in \mathscr{A}_\ell$ whose support is outside the ball of radius r is a negligible quantity. Then the above lemma implies that the difference between $\alpha_{\mathsf{F}}(t)(a)$ and its partial trace over all degrees of freedom outside the ball is also negligible.

Somewhat more generally, we may consider continuous maps $\mathsf{F} : [0,1] \to \mathfrak{D}_{al}$ and exponentiate them to a one-parameter family of automorphisms of $\mathscr{A}_{a\ell}$ satisfying

$$\frac{d\alpha_{\mathsf{F}}(t)(a)}{dt} = \alpha_{\mathsf{F}}(t)(\mathsf{F}(t)(a))$$

In the physics terminology, this is a path-ordered exponential of the time-dependent family of Hamiltonians $\mathsf{F}(t)$.

Definition 5. A Locally-Generated Automorphism is an automorphism of $\mathscr{A}_{a\ell}$ of the form $\alpha_{\mathsf{F}}(t)$ for some path $\mathsf{F} : [0,1] \to \mathfrak{D}_{al}$.

One can show that LGAs form a group. One can think of it as an infinite-dimensional Lie group whose Lie algebra is $\mathfrak{D}_{al}$.

Definition 6. Let G be a Lie group. A smooth action of G on a lattice system is a smooth homomorphism from G to the group of LGAs.

Given such an action, we also get an action on states of a lattice system: if ω is one state, then an LGA maps it to a new state $\omega' = \omega \circ \alpha$. That is, by definition, we have

$$\omega'(a) = \omega(\alpha(a)).$$

Clearly, since α is a $*$-automorphism, ω' is also a state.

8 Derivations which do not excite a state

Let ψ be any state.

Definition 7. A derivation $\mathsf{F} \in \mathfrak{D}_{al}$ does not excite ψ if for any $a \in \mathscr{A}_{a\ell}$ one has $\psi(\mathsf{F}(a)) = 0$.

To understand what this means, let's ask what happens to the state ψ when we act on it with a one-parameter group of LGAs generated by F. For any $a \in \mathscr{A}_{a\ell}$ we have

$$\frac{d}{dt}\psi(\alpha_{\mathsf{F}}(t)(a)) = \psi(\alpha_{\mathsf{F}}(t)(\mathsf{F}(a))) = \psi(\mathsf{F}(\alpha_{\mathsf{F}}(t)(a))) = 0.$$

Thus $\alpha_{\mathsf{F}}(t)$ preserves ψ and can be implemented in the GNS representation of ψ by a one-parameter group of unitaries $U(t)$ which preserve the GNS vacuum vector Ω. In other words, we have

$$\pi_\psi(\alpha_{\mathsf{F}}(t)(a)) = U(t)\pi_\psi(a)U(t)^{-1},$$

as well as $U(t)\Omega = \Omega$. Differentiating with respect to t and using Stone's theorem, we see that $\mathsf{F}\Omega = 0$. where $\hat{F}$ is an unbounded operator (the generator of the family $U(t)$). This unbounded operator represents the derivation F in the GNS representation. The fact that $\hat{F}$ annihilates the vacuum vector Ω means that F is an "unbroken symmetry" of ω.

It is easy to check that derivations which do not excite some particular state ψ form a Lie algebra (sub-algebra of $\mathfrak{D}_{al}$). Let's denote it $\mathfrak{D}^\psi_{al}$.

Similarly, we can define a p-chain f which does not excite the state ψ by

$$\psi([\mathsf{f}_{j_0\dots j_p}, a]) = 0$$

for all $a \in \mathscr{A}_{a\ell}$. It is easy to see that together they form a DGLA $C^\psi_\bullet$. It is a sub-DGLA of the DGLA $C_\bullet$.

What is the homology of this DGLA? In general, hard to tell. But we will now show that the homology is trivial for *gapped* states.

9 Gapped states

A gapped state is a state ψ such that there exists $\mathsf{H} \in \mathfrak{D}_{al}$ and a $\Delta > 0$ such that

$$-i\psi(a^*\mathsf{H}(a)) \geq \Delta(\psi(a^*a) - |\psi(a)|^2).$$

In particular, any gapped state is a ground state for H and thus is invariant under the 1-parameter group of automorphisms generated by H. Therefore we can rewrite the above equation as a condition on the Hamiltonian $\hat{H}$ in the GNS representation of ψ:

$$-i\langle\Omega, \pi_\psi(a)^\dagger[\hat{H}, \pi_\psi(a)]\Omega\rangle \geq \Delta\left(\langle\Omega, \pi_\psi(a)^\dagger\pi_\psi(a)\Omega\rangle - |\langle\Omega, \pi_\psi(a)\Omega\rangle|^2\right).$$

Since vectors of the form $\pi_\psi(a)\Omega$ are dense in the GNS Hilbert space, this inquality implies that in the orthogonal complement of Ω the unbounded operator $\hat{H}$ satisfies $-i\hat{H} \geq \Delta$. (We have a factor $-i$ because in our normalization Hamiltonians are anti-self-adjoint rather than self-adjoint.) So all states in the GNS Hilbert space orthogonal to Ω have energy at least Δ.

Theorem 2. If ψ is a gapped state, then the homology of the complex $\mathfrak{D}^\psi_{al}$ is trivial.

It is this property which makes possible the definition of topological invariants.