2020 - 10 - 16 Kähler geometry

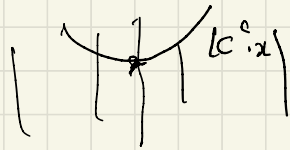
$L \to N$ ample line bundle over a compact complex mfd N. $\to$ alg.

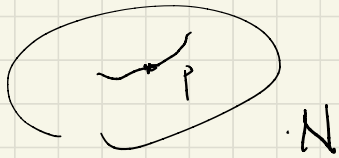
Assume K^c action on N lifts to L.

then also to $L^{-1} \leftarrow \mathcal{O}(-1)$. $\pi : L^{-1} \to N$

$p \in N$ is polystable $\underset{\text{def}}{\Longleftrightarrow}$ for $\overset{\neq 0}{x} \in \pi^{-1}(p)$

($K^c \cdot p$ is polystable) $K^c \cdot x$ is closed.





Kempf - Ness thm

$p \in N$ is polystable $\Longleftrightarrow$ $K^c \cdot p$ has a zero of the moment map $\mu : H \to k^*$.

Fujiki - Donaldson picture. applies Kempf-Ness to the infinite dimensional moment map geometry.

N = the space of Kähler forms $\left| \begin{array}{l} L \to M \\ c_1(L) \end{array} \right.$

$\cong$ the space of almost complex strs.

$\mu(\omega)$ = scalar curvature. as L^2-dual.

$\mu(\omega) = 0 \iff$ scal = const.

$N = \{\omega \mid \omega \in c_1(L)\}$

$\mathfrak{k} = \mathfrak{g} = \{u_X \in C^\infty(M) \mid$

X hol on M

$\int_M u_X \omega^n = 0 \}$

$\mu(\omega) \in \mathfrak{k}^* \ni$ scal

$$\langle \text{scal}, u \rangle_{L^2} = \int_M \text{scal} \cdot u \, \omega^n$$

If scal = const

$$\Rightarrow \text{RHS} = \text{scal} \int_M u_X \omega^n = 0.$$

$\mathcal{K}^c \cdot p \cong$ the space of Kähler forms

$\cap$ in a fixed Kähler class.

N.

Proof of Kempf-Ness

General fact (1) Hilbert-Mumford criterion.

Stability holds for K^c if and only if stability holds for every one-parameter subgroup of K^c (i.e. $\mathbb{C}^*$-action).

General fact (2)

Let h be a Hermitian metric on Λ.

Then there is a unique connection satisfying

$\cancel{\partial_x T - \partial_t \xi = [\xi, T]}$

(i) $\nabla h = 0$

(ii) connection form is type $(1,0)$.

(Chern connection).

Concretely, the connection θ and curvature Θ are given by

local frame e on U

$h(e,e) =: h_U$

$$\theta = h^{-1}\partial h = \partial \log h.$$

$\partial \log h_U$

$$\Theta = \bar\partial \theta = \bar\partial \partial \log h = -\partial\bar\partial \log h.$$

$$\left(c_1(\Lambda) = \left[\frac{i}{2\pi}\Theta\right] = \left[-\frac{i}{2\pi}\partial\bar\partial \log h_U\right]\right)$$

homework

show this class is indep of h.)

Λ ample $\Leftrightarrow$ $-\partial\bar\partial \log h_U$ is positive.

$\Longleftrightarrow$ $\partial\bar\partial h_U^{-1}$ is positive (Λ^{-1})

Based on (1), (2), let Γ be an orbit of the $\mathbb{C}^*$-action on Λ^{-1}.

choose a trivialization of Λ^{-1} as $U \times \mathbb{C}$.

Then Γ is of the form.

$$\Gamma|_U = \Gamma \cap (U \times \mathbb{C}) = \{ \underbrace{g(t)}_{\mathbb{C}^*} (p, w) \in U \times \mathbb{C} \}$$

$$(p(t), w(t))$$

$$\left.\frac{d}{dt}\right|_{t=0} g(t)(p, w) = \underbrace{X_p}_{\mathbb{N}} + \underbrace{f(p)\, w \frac{\partial}{\partial w}}_{\text{fiber}}$$

$$=: \tilde{X}_p \quad \text{on } \Lambda^{-1}$$

Let

$$H : \Gamma \ni \gamma \in \Lambda^{-1} \longrightarrow \mathbb{R}$$

be the function defined by

$$H(\gamma) = \log |\gamma|^2 = \log (h^{-1} w \bar{w}).$$

h_0 $\quad w \in \mathbb{C}^*$

Then

$$\partial H = -h^{-1} \partial h + \frac{dw}{w}$$

$$\left(w \in \Lambda^{-1}, \quad z = \frac{1}{w} \in \Lambda \right) \quad \text{on } \Lambda^*$$

$$\rightsquigarrow \partial H = -h^{-1} \partial h - \frac{dz}{z} = -\tilde{\theta} \quad \text{on } (\Lambda^{-1})^*$$

we studied last time

$$u_X = \tilde{\theta}(\tilde{X}) = -XH$$

So, H has a critical point along Γ if and only if $u_X = 0$ somewhere on Γ.

Moreover H is a convex function

$$\because \quad i\partial\bar{\partial} H = i\partial\bar{\partial} \log |\gamma|^2 = -i\partial\bar{\partial} \log h$$

$$= i\Theta > 0. \quad \text{since } \Lambda \text{ ample.}$$

$\because$

Thus Λ_x has a zero along P

$\Longleftrightarrow$ moment map has a zero along P

$\Downarrow$

The convex function H has a critical point

$\Longleftrightarrow$ P is closed in Λ^{-1} - zero section.

$\Longleftrightarrow$ the orbit is closed.

<u>Summary</u>

H proper $\Longleftrightarrow$ the orbit is polystable in Λ^{-1}

H bounded from below $\Longleftrightarrow$ semi stable

H not bounded from below $\Longleftrightarrow$ unstable.

Hermitian Yang-Mills (Kobayashi - Hitchin)

$\rightsquigarrow$ H is Donaldson function.

cscK $\rightsquigarrow$ H is K-energy.

Donaldson–Fujiki picture

Scalar curvature and moment map.

"GIT-symplectic" stability

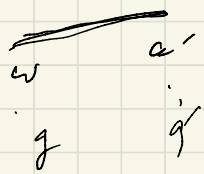
M compact Kähler manifold.

ω_0 fixed Kähler form.

$$\Omega = \{ \omega = \omega_0 + i\partial\bar\partial\varphi \mid \omega \text{ positive} \}.$$

$$\Uparrow$$

$$[\omega_0].$$



Moser's theorem

Let M be a compact symplectic mfd, and ω_t be a path by symplectic forms joining ω and ω'. Then $\exists$ diffeo $f : M \to M$ such that $\omega = f^*\omega'$

We apply this to two Kähler forms in a fixed cohomology class.

$$(M, \omega, J), \quad (M, \omega', J) \quad J'$$

$$f^*(M, \omega', J) = (M, \omega, f^*J)$$

Ordinary Kähler geometry (Calabi's setup)

J fixed, ω varying.

Donaldson–Fujiki picture (using Moser's thm)

ω fixed, J varying.

M compact symplectic mfd.

$$\dim_{\mathbb{R}} = n = 2m$$

ω fixed symplectic form.

$$Z = \left\{ J \in \mathbb{P}(\mathrm{End}(TM)) \;\middle|\; \begin{array}{l} \omega\text{-compatible} \\ \text{integrable almost} \\ \text{complex structure} \end{array} \right\}$$

$\overset{\shortparallel}{N}$ before.

<u>Def</u> ω-compatible

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \cdot\ \omega(JX, JY) = \omega(X, Y) \\ \cdot\ J^2 = -\mathrm{id} \\ \cdot\ g_J(X, Y) = \omega(X, JY) \text{ is a Kähler metric.} \end{cases}$$

$\Longleftrightarrow$ (M, ω, J) is a Kähler mfd.

$J \in Z$ fixed.

$$T^*M \otimes \mathbb{C} = T_J^{*\prime} M \oplus T_J^{*\prime\prime} M$$

i-eigensp $\quad (-i)$-eigensp of J.

$$T_J^{*\prime\prime} M = \overline{T_J^{*\prime} M}.$$

$J' \in Z$ another complex structure

$$T_J^* M \otimes \mathbb{C} = T_{J'}^{*\prime} M \oplus T_{J'}^{*\prime\prime} M$$

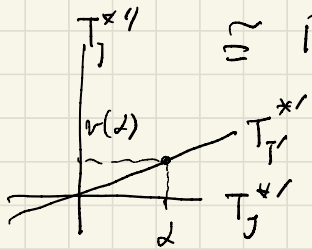
If J' is sufficiently close to J, then $T_{J'}^{*\prime} M$ is expressed as a graph over $T_J^{*\prime} M$ as

$$T_{J'}^{*\prime} M = \{ \alpha + v(\alpha) \mid \alpha \in T_J^{*\prime} M \}$$

where $v \in \Gamma(\mathrm{End}(T_J^{*\prime} M, T_J^{*\prime\prime} M))$

$$\cong \Gamma(T_J^{\prime} M \otimes T_J^{*\prime\prime} M)$$

$T^*M \otimes \mathbb{C}$



$$v \in \Gamma(T'_J M \otimes T''^{*}_J M) \underset{\text{metric}}{\cong} \Gamma(T'_J M \otimes T'_J M)$$

<u>Lemma</u> $v \in \underbrace{\mathrm{Sym}}_{\text{symmetric part}}(T'_J M \otimes T'_J M)$.