Gromov-Witten invariants: discussion

We briefly discuss calculation and significance of Gromov-Witten invariants, before using the same techiques which we used to define them to study Lagrangian Floer homology: the latter is a key ingredient in defining A-side categories in mirror symmetry.

<u>Note</u> I continue to follow viewpoint of Joyce.

<u>Question</u>: why do we call them "invariants"?

<u>Answer</u>: given a Kahler manifold (M, J, ω), we expect the invariants $\langle I_{g,m,\beta} \rangle$ to be unchanged by continuous variation of complex structure J.

Note: This fact was implicit in our construction of the $\langle I_{g,m,\beta} \rangle$ (as well as the $I_{g,m,\beta}$), as we used that the virtual classes VC are in fact independent of the almost complex structure J. This follows from work of Fukaya–Ono.

Axioms for GW invariants

The deformation invariance, and other properties of the invariants are formalized in a set of axioms. This approach originated with work of Kontsevich–Manin. These allow the invariants to be calculated algorithmically in many cases.

Rem For details, see [CK, §7.3.1], and for calculations for $M = \mathbb{P}^2$, see [CK, §7.4.2].

Quantum cohomology

A key application of GW invariants is to define a certain ring which we denote $QH^*(M,\mathbb{C})$, the multiplication on which is a deformation of that on $H^*(M,\mathbb{C})$. This structure, which has a string theory interpretation, turns out to corresponds to a certain "variation of Hodge structure" on the mirror M'. This was famously used by Candelas, de la Ossa, Green and Parkes to conjecture counts of curves on the quintic 3-fold (in our language, GW invariants.) For further details, see [CK, §8].

Morse theory

This provides conceptual motivation for the construction of Lagrangian Floer homology.

Setting: Take M compact smooth manifold, with a Morse function $f: M \to \mathbb{R}$, namely where $\mathrm{Crit}(f) = \{z_1, \ldots, z_N\}$ (finite, isolated critical points) and the Hessian $\mathrm{Hess}_{z_i}(f)$ is non-degenerate.

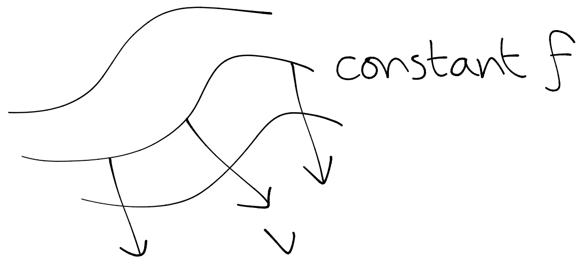
It follows that, in a neighbourhood U of each z_i, we have coordinates $\underline{x} = (x_1, \ldots, x_n)$ such that $z_i \leftrightarrow \underline{0}$ and

$$\boxed{f(\underline{x}) = f(z_i) - x_1^2 - \ldots - x_\mu^2 + x_{\mu+1}^2 + \ldots + x_n^2}$$

where μ depends on i, and is known as the Morse index of z_i

<u>Ex</u> $\mu = 0$ (n) for local minimum (maximum) of f.

Now choose a metric g on M, have vector field v such that $g(v, -) = -df$



<u>Def</u> flow lines for v are

$\gamma : \mathbb{R} \to M$ smooth with $d\gamma/dt = v|_{\gamma(t)}$

Then by compactness $\gamma(\pm) = \lim_{t \to \pm\infty} \gamma(t)$ exists,

and $\gamma(\pm) \in \{z_1, \ldots, z_N\}$

<u>Def</u> For distinct i, j we put

$$\mathcal{M}(z_i, z_j) = \{\gamma \mid \gamma(-) = z_i, \gamma(+) = z_j\} / \mathbb{R}$$

with $c \in \mathbb{R}$ acting by $(c \cdot \gamma)(t) = \gamma(t + c)$

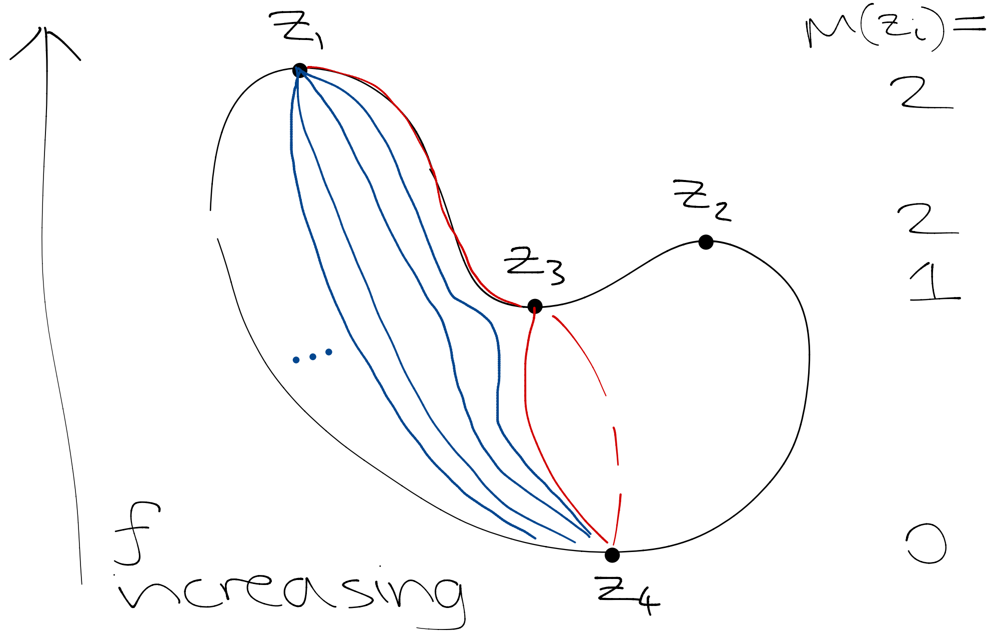
If g is chosen to be generic, then $\mathcal{M}(z_i, z_j)$ is smooth, compact and oriented, with

$$\dim \mathcal{M}(z_i, z_j) = \mu(z_i) - \mu(z_j) - 1$$

and boundary $\partial \mathcal{M}(z_i, z_j) = \coprod_{k \neq i,j} \mathcal{M}(z_i, z_k) \times \mathcal{M}(z_k, z_j)$ (*)

Ex For $M \cong S^2$ and f as shown, we see that $\mathcal{M}(z_1, z_4) \cong I$ with boundary

$$\mathcal{M}(z_1, z_3) \times \mathcal{M}(z_3, z_4)$$
$$= pt \times 2pt = 2pt$$



Morse homology

Def (Morse complex) Let $C_\ell = \langle z_i \mid \mu(z_i) = \ell \rangle_{\mathbb{R}}$ and define $\partial : C_\ell \to C_{\ell-1}$ by letting

$$\partial z_i = \sum_{\mu(z_j) = \ell - 1} \# \mathcal{M}(z_i, z_j) \cdot z_j$$

Note The $\mathcal{M}(z_i, z_j)$ appearing here is 0-dimensional, so $\#$ means "count points, with signs from the orientation".

∂ is a differential. The (z_i, z_j) matrix element of ∂^2 is

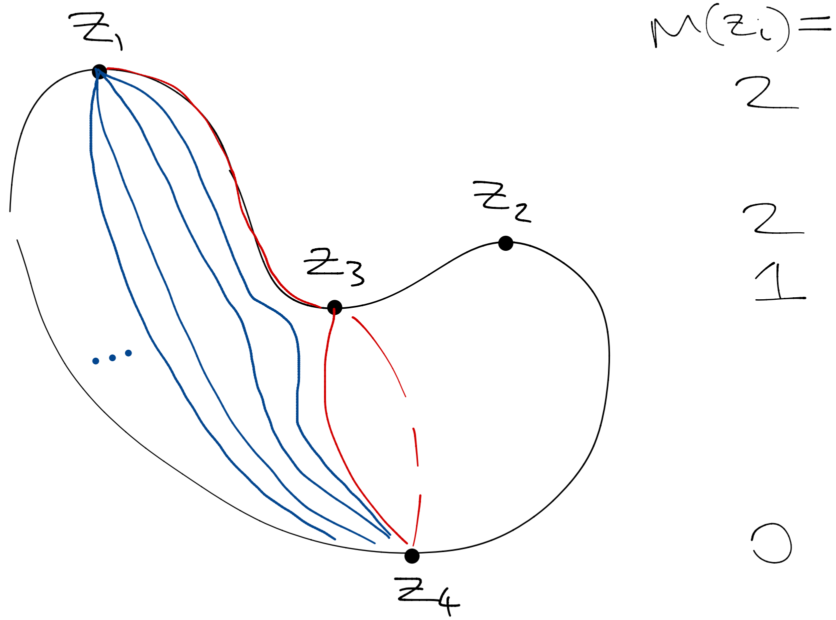
$$\sum_{\mu(z_k) = \ell - 1} \underbrace{\# \mathcal{M}(z_i, z_k) \, \# \mathcal{M}(z_k, z_j)}_{= \# \partial \mathcal{M}(z_i, z_j) = 0}$$

Def (Morse homology) $H_*^{Mor}(M) = H_*(C_*, \partial)$

Note This is independent of f and metric g, and is isomorphic to homology defined in other ways.

Ex $\mathcal{M}(z_3, z_4) = 2pt,$

but $\#\mathcal{M}(z_3, z_4) = +1 - 1 = 0$



The complex (C_*, ∂) is given by

$$\mathbb{R}^2 \xrightarrow{(1\ 1)} \mathbb{R} \xrightarrow{0} \mathbb{R}$$

So H_* gives homology of S^2, 1-dim in degrees 0 and 2.

Lagrangian Floer homology

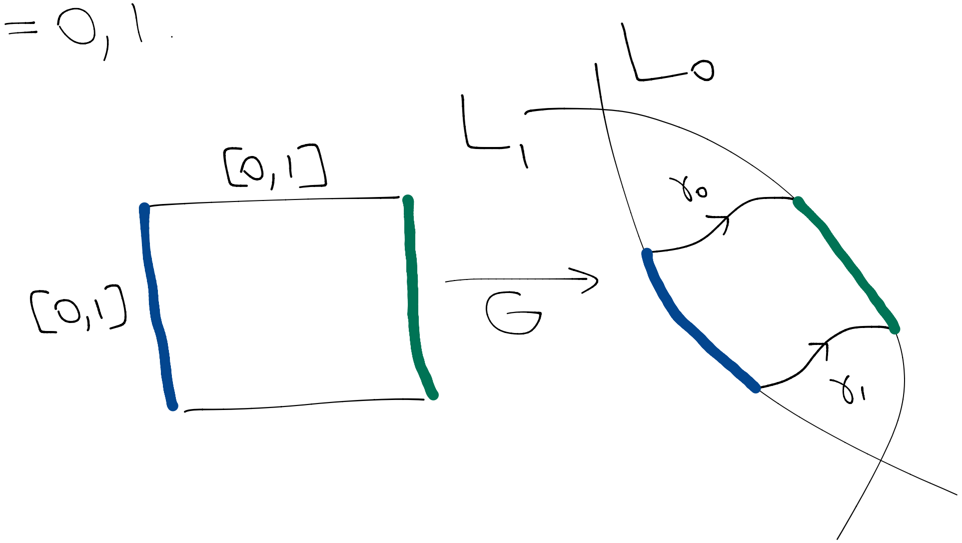
Setting (M, ω) symplectic manifold, L_0, L_1 Lagrangians (compact, connected)

with $L_0 \cap L_1 = \{p_1 - p_N\}$, each a transverse intersection

Def $\mathcal{P} = \mathcal{P}(L_0, L_1) = \{\text{paths from } L_0 \text{ to } L_1\}$

$$= \{\gamma : [0,1] \to M \text{ smooth} \mid \gamma(i) \in L_i \;\; i = 0,1\}$$

Choose a basepoint $\gamma_0 \in \mathcal{P}$. Then given a further $\gamma_1 \in \mathcal{P}$, a path $\{\gamma_t\}_{t\in[0,1]}$ in $\mathcal{P}$ joining γ_0 and γ_1 may be viewed as $G : [0,1]^2 \to M$ defined by $G(s,t) = \gamma_t(s)$. Observe then that $G(\{i\} \times [0,1]) \subset L_i$ for $i = 0,1$.



We want to define $F : \mathcal{P}(L_0, L_1) \to \mathbb{R}$

$$\gamma_1 \mapsto \int_{[0,1]^2} G^*(\omega)$$

and treat it like a Morse function on $\mathcal{P}$ to obtain a cohomology theory.

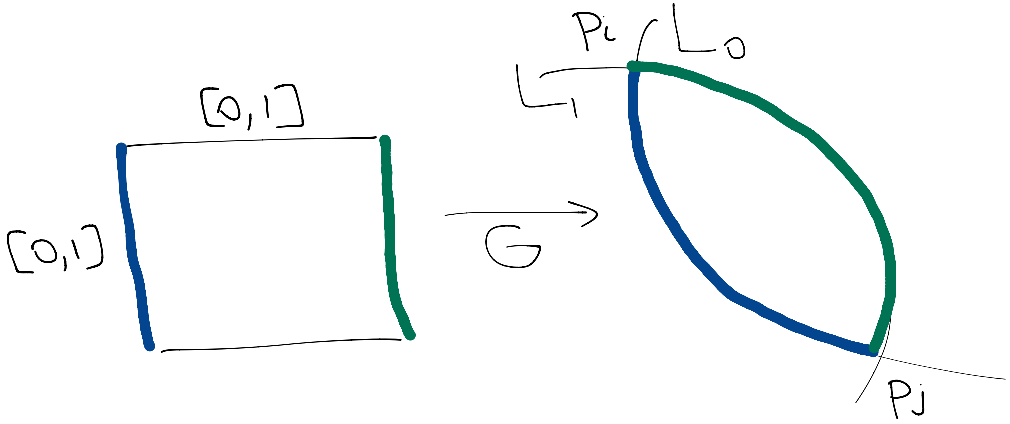
There are two immediate problems however

1. the integral may depend on the choice of path γ_t
2. $\mathcal{P}$ is infinite dimensional

The first may be addressed by replacing $\mathcal{P}$ with its universal cover. To address the second, we work out a sequence of analogies, before defining our cohomology theory.

Lagrangian intersection / Morse theory analogy

A constant path $\gamma_i(s) \equiv p_i$ corresponds to a critical point of F. A path $\{\gamma_t\}$ between such paths for, say, p_i and p_j then corresponds to a map G pictured as follows



The condition of γ_t being a flow line associated to F corresponds to G being a J-holomorphic curve.

By analogy with Morse theory, we therefore construct our differential for Lagrangian Floer homology by "counting" such curves.