Combinatorics Lecture 13 2022/7/12

§1. Bollobás' Theorem

<u>Thm 1</u>. Let $A_1, A_2, \dots, A_m$ and $B_1, B_2, \dots, B_m$ be subsets of some ground set Ω. If we have

(1) $A_i \cap B_i = \phi$ for any $i \in [m]$,

(2) $A_i \cap B_j \neq \phi$, for any $i \neq j \in [m]$,

then
$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

<u>pf</u>. Let $X = \bigcup_{i=1}^{m} (A_i \cup B_i)$ and $|X| = n$.

We will prove by induction on n.

- <u>Base case $n=1$</u>: $A_1 = \{1\}$, $B_1 = \phi$
- Now we assume that the statement holds for $|X| \leq n-1$. For $x \in X$,

let $I_x = \{ i \in [m] : x \notin A_i \}$

Define $\mathcal{F}_x = \{A_i : i \in I_x\} \cup \{B_i \setminus \{x\} : i \in I_x\}$.

Since each set in $\mathcal{F}_x$ doesn't contain x,

$$\left| \bigcup_{S \in \mathcal{F}_x} S \right| \leq |X \setminus \{x\}| = n-1.$$

Also, it is easy to see that these sets in $\mathcal{F}_x$ satisfy the induction hypothesis. Then by induction

$$\sum_{i \in I_x} \frac{1}{\binom{|A_i| + |B_i \setminus \{x\}|}{|A_i|}} \leq 1, \quad \forall x \in X.$$

We sum up all $x \in X$ to get

$$(*) \quad \sum_{x \in X} \sum_{i \in I_x} \frac{1}{\binom{|A_i| + |B_i \setminus \{x\}|}{|A_i|}} \leq n.$$

For each $i \in [m]$, it contribues either 0, $\frac{1}{\binom{|A_i| + |B_i|}{|A_i|}}$, or $\frac{1}{\binom{|A_i| + |B_i| - 1}{|A_i|}}$ for each $x \in X$.

The term $\frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}$ occurs when $i \in I_x$ & $x \notin B_i$

$\iff x \notin A_i \cup B_i$

while the term $\frac{1}{\binom{|A_i|+|B_i|-1}{|A_i|}}$ occur when

$i \in I_x$ & $x \in B_i \iff x \in B_i$.

Then we can rewrite $(*)$ as

$$\sum_{i=1}^{m}\left((n-|A_i|-|B_i|)\frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} + |B_i|\frac{1}{\binom{|A_i|+|B_i|-1}{|A_i|}}\right) \leq n$$

$$\iff \sum_{i=1}^{m}\frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$$

<u>Corollary</u> (LYM-inequality) Let $\mathcal{A} \subseteq 2^{[n]}$ be a family such that $\forall A, B \in \mathcal{A}$, $A \nsubseteq B$ and $B \nsubseteq A$.

Then $$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \le 1.$$

Pf: Exercise (via Bollobás' Thm). ☐

Corollary (Sperner's Thm) Under the same family $\mathcal{A}$, $$|\mathcal{A}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Thm 2 (The skew version of Bollobás' Thm)

Let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be sets such that

(1) $A_i \cap B_i = \emptyset$, $\forall i \in [m]$

(2) $A_i \cap B_j \neq \emptyset$, $\forall i < j$.

If $|A_i| = r$ for $\forall i \in [m]$ and $|B_j| = s$ for $\forall j \in [m]$, then $$m \le \binom{r+s}{r}.$$

Def. Let $\mathbb{F}$ be a field. A set $A \subseteq \mathbb{F}^n$

is in general position, if any n vectors in A are linearly independent over $\mathbb{F}$.

$\boxed{\text{Fact}}$. For $a \in \mathbb{F}$, let $\vec{m}(a) = (1, a, a^2, \dots, a^{n-1}) \in \mathbb{F}^n$ be the moment curve. Then

$A = \{ \vec{m}(a) : a \in \mathbb{F} \}$ is in general position

(because of the Vandermonde matrix)

pf of Thm 2. Take a set V of vectors $\vec{v} = (v_0, v_1, \dots, v_r) \in \mathbb{R}^{r+1}$ such that

- V is in general position, and
- $|V| = \left| \bigcup_{i \in [m]} (A_i \cup B_i) \right|$.

Then we identify the elements of $\bigcup_{i \in [m]} (A_i \cup B_i)$ with the vectors in V. So we will view

A_i or B_j as a subset in $V \subseteq \mathbb{R}^{r+1}$,

where $|A_i| = r$ and $|B_j| = s$.

For each $j \in [m]$, we define

$$f_j(\vec{x}) = \prod_{\vec{v} \in B_j} \vec{x} \cdot \vec{v}, \quad \text{where } \vec{x} = (x_0, \cdots, x_r) \in \mathbb{R}^{r+1}.$$

So $f_j(\vec{x})$ is generated by the following

monomials $x_0^{i_0} x_1^{i_1} \cdots x_r^{i_r}$, $\sum_{j=0}^{r} i_j = s$ & $i_j \geq 0$

There are exactly $\binom{r+s}{r}$ such monomials.

$\Rightarrow$ $f_1, f_2, \cdots, f_m$ are contained in a

polynomial linear space of dimension $\binom{r+s}{r}$.

To show $\boxed{m \leq \binom{r+s}{r}}$, it suffices to prove

that $f_1, \cdots, f_m$ are linearly independent.

Note that

$f_j(\vec{x}) = 0$ iff $\exists \vec{v} \in B_j$ with $\vec{x} \cdot \vec{v} = 0$. ①

Consider the linear subspace $\mathrm{Span}(A_j) \subseteq \mathbb{R}^{r+1}$,

where $|A_j| = r$.

Since $A_j \subseteq V$ and V is in general position,

we see that all r vectors in A_j are linear indep.

$$\Rightarrow \quad \dim(\mathrm{Span}(A_j)) = r$$

$$\Rightarrow \quad \dim\left((\mathrm{span}(A_j))^{\perp} \right) = 1$$

There exists $\vec{a_j} \in (\mathrm{Span}(A_j))^{\perp}$ for $\forall j \in [m]$.

For each $\vec{v} \in V$, ②

$$\vec{v} \cdot \vec{a_j} = 0 \text{ iff } \vec{v} \in \mathrm{span}(A_j) \text{ iff } \vec{v} \in A_j.$$

This is because if $\vec{v} \in V \setminus A_j$, then all $r+1$ vectors in $A_j \cup \{\vec{v}\}$ are linearly indep. (as V is in general position)

which contradicts that $\vec{v} \in \mathrm{Span}(A_j)$.

Combining ① and ②,

$$f_j(\vec{a_i}) = \prod_{\vec{v} \in B_j} \vec{a_i} \cdot \vec{v} = 0 \quad \text{iff}_{①}$$

$$\exists\ \vec{v} \in B_j \text{ with } \vec{a_i} \cdot \vec{v} = 0 \quad \text{iff}_{②}$$

$$\exists\ \vec{v} \in A_i \cap B_j .$$

By condition (2) that $A_i \cap B_j \neq \emptyset$ for $i < j$, we have

$$\begin{cases} f_j(\vec{a_i}) = 0 & \text{for } \forall\, i < j \\ f_i(\vec{a_i}) \neq 0 & \text{for } \forall\, i \end{cases}$$

By the previous lemma, $f_1, \dots, f_m$ are indeed linearly independent. $\boxtimes$

§ 2. Hoffman's bound

Def. The adjacency matrix $A_G = (a_{ij})_{n\times n}$ of an n-vertex graph G is defined by

$$a_{ij} = \begin{cases} 1, & ij \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

So A_G is 0/1 symmetric $n\times n$ matrix

$\Rightarrow$ All its eigenvalues are reals.

Def. The eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ of A_G are called the eigenvalues of G. The eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ of A_G satisfying

$$\begin{cases} A_G \vec{v}_i = \lambda_i \vec{v}_i \\ \|\vec{v}_i\| = 1 \\ \vec{v}_i \perp \vec{v}_j \text{ for } \forall i \neq j \end{cases}$$

are called orthonormal eigenvectors of G.

Thm 3 (Hoffman's bound) If an n-vertex graph G is d-regular with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, then $\boxed{\alpha(G) \leq n \frac{-\mu_n}{d-\mu_n}}$

where $\alpha(G)$ denotes the maximum size of an independent set in G.

Exercise: If G is d-regular, then $\mu_1 = d$.

Pf: Let $V(G) = [n]$. Let $\vec{v}_1, \ldots, \vec{v}_n$ be the orthonormal eigenvectors of G.

Let I be the maximum independent set in G,

so $\alpha(G) = |I|$.

Let $\vec{\mathbb{1}}_I \in \{0,1\}^n$ be such that

$$\vec{\mathbb{1}}_I(j) = \begin{cases} 1, & j \in I \\ 0, & \text{otherwise}. \end{cases}$$

Let $\vec{1}_I = \sum_{i=1}^{n} \alpha_i \vec{v_i}$.

where $\alpha_i = \vec{1}_I \cdot \vec{v_i}$.

Consider $\vec{1}_I^{\,T} \cdot A_G \cdot \vec{1}_I = \sum_{1 \le i,j \le n} \vec{1}_I(i) \cdot a_{ij} \cdot \vec{1}_I(j)$

$$= 0$$

Then

$$0 = \vec{1}_I^{\,T} A_G \vec{1}_I$$

$$= \left(\sum_i \alpha_i \vec{v_i}\right)^T A_G \left(\sum_j \alpha_j \vec{v_j}\right)$$

$$= \sum_{i,j} \alpha_i \alpha_j \vec{v_i}^T A_G \vec{v_j}$$

$$= \sum_{i=1}^{n} \mu_i \alpha_i^2$$

$$\geq \mu_1 \alpha_1^2 + \mu_n(\alpha_2^2 + \cdots + \alpha_n^2) \quad (3)$$

Also

$$\alpha(G) = |I| = \vec{1}_I \cdot \vec{1}_I$$

$$= \langle \sum \alpha_i \vec{v_i}, \sum \alpha_i \vec{v_i} \rangle = \sum_{i=1}^{n} \alpha_i^2 \quad (1)$$

Also $\mu_1 = d$ and $\vec{v}_1 = \vec{\mathbb{1}}/\sqrt{n}$

$$\Rightarrow \quad \alpha_1 = \vec{\mathbb{1}}_I \cdot \vec{v}_1 = \frac{|I|}{\sqrt{n}}$$

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$$\Longrightarrow \quad 0 \geq \frac{|I|^2}{n}\mu_1 + \left(|I| - \frac{|I|^2}{n}\right)\mu_n$$

$$\Longrightarrow \quad \alpha(G) = |I| \leq n \cdot \frac{-\mu_n}{\mu_1 - \mu_n} \qquad \blacksquare$$

Def. A Kneser graph $K(n,k)$ with $n \geq 2k$ is a graph with vertex set $\binom{[n]}{k}$ such that for any two sets $A, B \in \binom{[n]}{k}$, A is adjacent to B if and only if $A \cap B = \emptyset$.

Fact. Any independent set of $K(n,k)$ is an intersecting family in $\binom{[n]}{k}$.

Erdős–Ko–Rado Thm (Restatement)

$$\alpha(K(n,k)) \leq \binom{n-1}{k-1}.$$

Pf: We need the following fact that

the eigenvalues of $K(n,k)$ are:

$$\mu_j = (-1)^j \binom{n-k-j}{k-j} \text{ of multiplicity } \binom{n}{j} - \binom{n}{j-1}$$

for $j = 0, 1, \dots, k$.

Since $K(n,k)$ is d-regular with $d = \binom{n-k}{k}$,

by Hoffman's bound, we get

The max eigen. is d, while the min eigen is $\mu_1 = -\binom{n-k-1}{k-1}$

$$\alpha(K(n,k)) \leq \binom{n}{k} \cdot \frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k} + \binom{n-k-1}{k-1}}$$

$$= \binom{n-1}{k-1}$$

Def. The Rayleigh quotient of a vector $\vec{x} \in \mathbb{R}^n$ with respect to an $n \times n$ matrix M is defined

$$\frac{\vec{x}^T M \vec{x}}{\vec{x}^T \vec{x}}.$$

Thm 4. Let M be a symmetric matrix and Let $\vec{x} \in \mathbb{R}^n$ be a non-zero vector that maximize the Rayleigh quotient $\frac{\vec{x}^T M \vec{x}}{\vec{x}^T \vec{x}}$.

Then this Rayleigh quotient $\lambda = \frac{\vec{x}^T M \vec{x}}{\vec{x}^T \vec{x}}$ is the largest eigenvalue of M and $\vec{x}$ is an eigenvector of λ.

Thm 5. (Courant-Fischer Thm) Let L be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$\lambda_k = \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \max_{\vec{x} \in S} \frac{\vec{x}^T L \vec{x}}{\vec{x}^T \vec{x}}$$

$$= \max_{\substack{T \subseteq \mathbb{R}^n \\ \dim(T)=n-k+1}} \min_{x \in T} \frac{\vec{x}^T L \vec{x}}{\vec{x}^T \vec{x}}.$$

<u>Def</u>: The <u>Laplacian matrix</u> L_G of a graph G is defined by $L_G = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} - A_G$,

where d_i denotes the degree of the vertex i.

<u>Thm 6</u> (Godsil-Newman) Let S be an independent set in G, and let

$$d_{ave}(S) = \frac{\sum_{i \in S} d_i}{|S|}.$$

Then $|S| \leq n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right)$, where $n = |V(G)|$ and λ_n is the maximum eigenvalue of the matrix L_G.

Remark Thm 6 $\Rightarrow$ Thm 4.

Let G be d-regular. $\Rightarrow d_{ave}(S) = d$

& $\lambda_n = d - \mu_n$, where μ_n is the least eigenvalue of A_G

Then
$$|S| \le n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right)$$
$$= n\left(1 - \frac{d}{d-\mu_n}\right) = n\,\frac{-\mu_n}{d-\mu_n}.$$

pf of Thm 6. Consider L_G.

Then
$$\lambda_n = \max_{\vec{x}} \frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}} \quad (\text{by Thm 4})$$

$$\Rightarrow \lambda_n = \max_{\vec{x} \perp \vec{1}} \frac{\vec{x}^T L_G \vec{x}}{\vec{x}^T \vec{x}} \quad \checkmark$$

Consider $\vec{x_S} = \vec{\mathbb{1}}_S - s\cdot\vec{\mathbb{1}}$,

where $s = \frac{|S|}{n}$. $\quad (\vec{x_S} \perp \vec{\mathbb{1}})$

$$\Rightarrow \quad \lambda_n \geq \frac{\vec{x_S}^T L_G \vec{x_S}}{\vec{x_S}^T \vec{x_S}},$$

where $\vec{x_S}^T L_G \vec{x_S} = \sum_{i\in S} d_i = d_{ave}(S)\cdot|S|$.

and $\vec{x_S}^T \vec{x_S} = n(s-s^2)$.

$$\Rightarrow \quad \lambda_n \geq \frac{d_{ave}(S)\cdot|S|}{n(s-s^2)} = \frac{d_{ave}(S)}{1-s}$$

$$\Rightarrow \quad 1 - \frac{d_{ave}(S)}{\lambda_n} \geq s = \frac{|S|}{n}. \quad \boxtimes$$

Thm 7 (Hoffman bound) For any G

(not necessarily regular), $\chi(G) \geq 1 + \frac{\mu_1}{-\mu_n}$

Thm 8

$$\chi(G) \leq 1 + \lfloor \mu_1 \rfloor .$$