Biased random walk on G-W tree

Edge local time and heavy range in the recurrent case

Convergences of heavy range: ideas of proof

Heavy range and edge local times of biased random walk on trees

Xinxin Chen[1]

[1]University of LYON 1, France
Joint work with P. Andreoletti and L. de Raphélis (Orléans)

May 2021

Table of Contents

1 Biased random walk on G-W tree

2 Edge local time and heavy range in the recurrent case

3 Convergences of heavy range: ideas of proof

Biased random walk on supercritical G-W tree $\mathbb{T}$

Given $(A_u, u \in \mathbb{T})$,

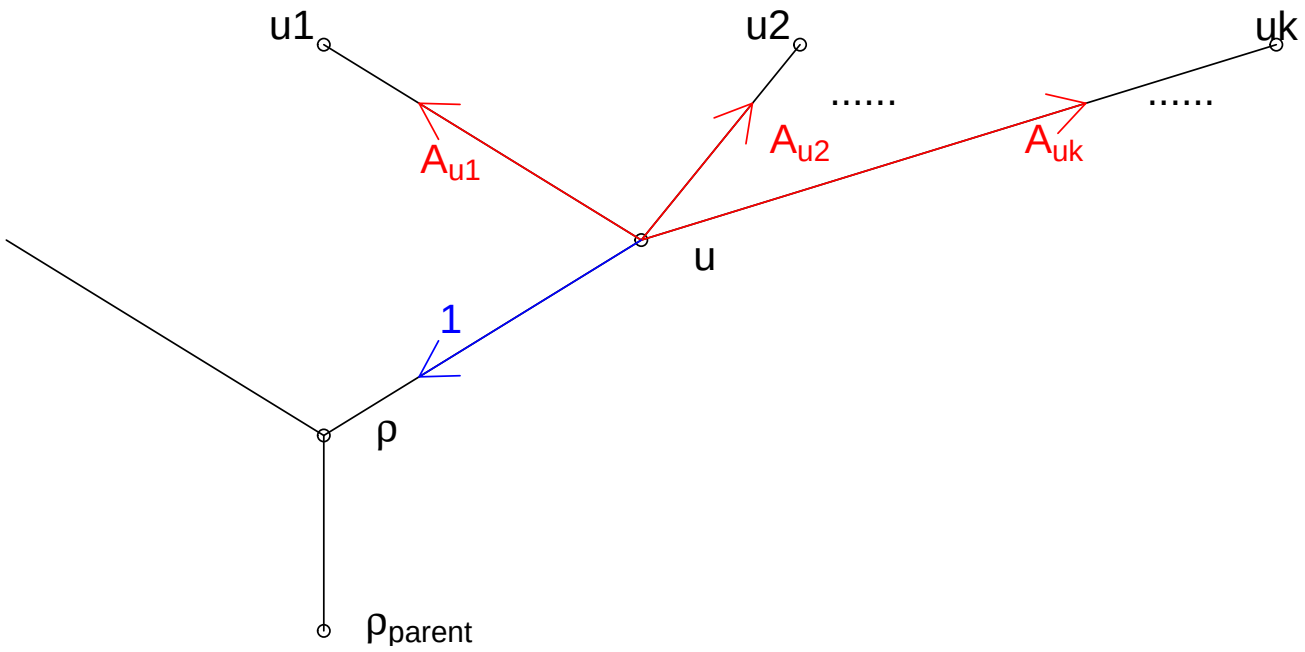

Figure: Biased random walk with transition probabilities: $p(u \to uk) \propto A_{uk}$ and $p(u \to u_{parent}) \propto 1$. $N_u =$ nb of children of u

$$\forall u \in \mathbb{T}, (A_{u1}, \cdots, A_{uN_u}) \text{ are i.i.d. copies of point process } \mathcal{L}.$$

Random environment: $(A_u, u \in \mathbb{T})$

- $|u| = dist_{\mathbb{T}}(\rho, u)$.
- $\rho \to u_1 \to u_2 \to \cdots \to u_{|u|} = u$: ancestral line of u.
- $V(u) := -\sum_{k=1}^{|u|} \log A_{u_k}$, $V(\rho) = 0$.

Random environment: $(A_u, u \in \mathbb{T})$

- $|u| = dist_{\mathbb{T}}(\rho, u)$.
- $\rho \to u_1 \to u_2 \to \cdots \to u_{|u|} = u$: ancestral line of u.
- $V(u) := -\sum_{k=1}^{|u|} \log A_{u_k}$, $V(\rho) = 0$.
- $$p(X_{n+1} = v | X_n = u) = \begin{cases} \frac{e^{-V(v)} \mathbf{1}_{\{v \text{ is child of } u\}}}{e^{-V(u)} + \sum_{z:\text{ child of } u} e^{-V(z)}} \\ \frac{e^{-V(u)} \mathbf{1}_{\{v \text{ is parent of } u\}}}{e^{-V(u)} + \sum_{z:\text{ child of } u} e^{-V(z)}} \end{cases}$$

  with $p(X_{n+1} = \rho | X_n = \rho_{parent}) = 1$.

Random environment: $(A_u, u \in \mathbb{T})$

- $|u| = dist_{\mathbb{T}}(\rho, u)$.
- $\rho \to u_1 \to u_2 \to \cdots \to u_{|u|} = u$: ancestral line of u.
- $V(u) := -\sum_{k=1}^{|u|} \log A_{u_k}$, $V(\rho) = 0$.
- $$p(X_{n+1} = v | X_n = u) = \begin{cases} \frac{e^{-V(v)} \mathbf{1}_{\{v \text{ is child of } u\}}}{e^{-V(u)} + \sum_{z:\text{ child of } u} e^{-V(z)}} \\ \frac{e^{-V(u)} \mathbf{1}_{\{v \text{ is parent of } u\}}}{e^{-V(u)} + \sum_{z:\text{ child of } u} e^{-V(z)}} \end{cases}$$

  with $p(X_{n+1} = \rho | X_n = \rho_{parent}) = 1$.
- $(V(u), u \in \mathbb{T})$ is branching random walk.

$$\Psi(t) := \log \mathbf{E}\left[\sum_{|u|=1} e^{-tV(u)}\right], \Psi(0) = \log \mathbf{E}[N] > 0.$$

Criterion of Recurrence/Transience on $\mathbb{T} = \infty$

[Lyons-Pemantle'92]

- Transience: $\inf_{t\in[0,1]} \Psi(t) > 0$.

Criterion of Recurrence/Transience on $\mathbb{T} = \infty$

[Lyons-Pemantle'92]

- Transience: $\inf_{t\in[0,1]} \Psi(t) > 0$.
- Recurrence: $\inf_{t\in[0,1]} \Psi(t) \leq 0$, [Faraud'11]

$$\text{Positive recurrence} \begin{cases} \inf_{t\in[0,1]} \Psi(t) < 0; \\ \inf_{t\in[0,1]} \Psi(t) = 0, \Psi'(1) > 0; \end{cases}$$

$$\text{let} t_0 := \inf\{t \in [0,1] : \Psi(t) = 0\}.$$

$$\text{Null recurrence} \begin{cases} \inf_{t\in[0,1]} \Psi(t) = 0, \Psi'(1) = 0; \\ \inf_{t\in[0,1]} \Psi(t) = 0, \Psi'(1) < 0, \text{ let} \\ \kappa := \inf\{t > 1; \Psi(t) = 0\} \in (1, \infty]. \end{cases}$$

Recurrent case: $\inf_{t\in[0,1]} \Psi(t) \leq 0$.

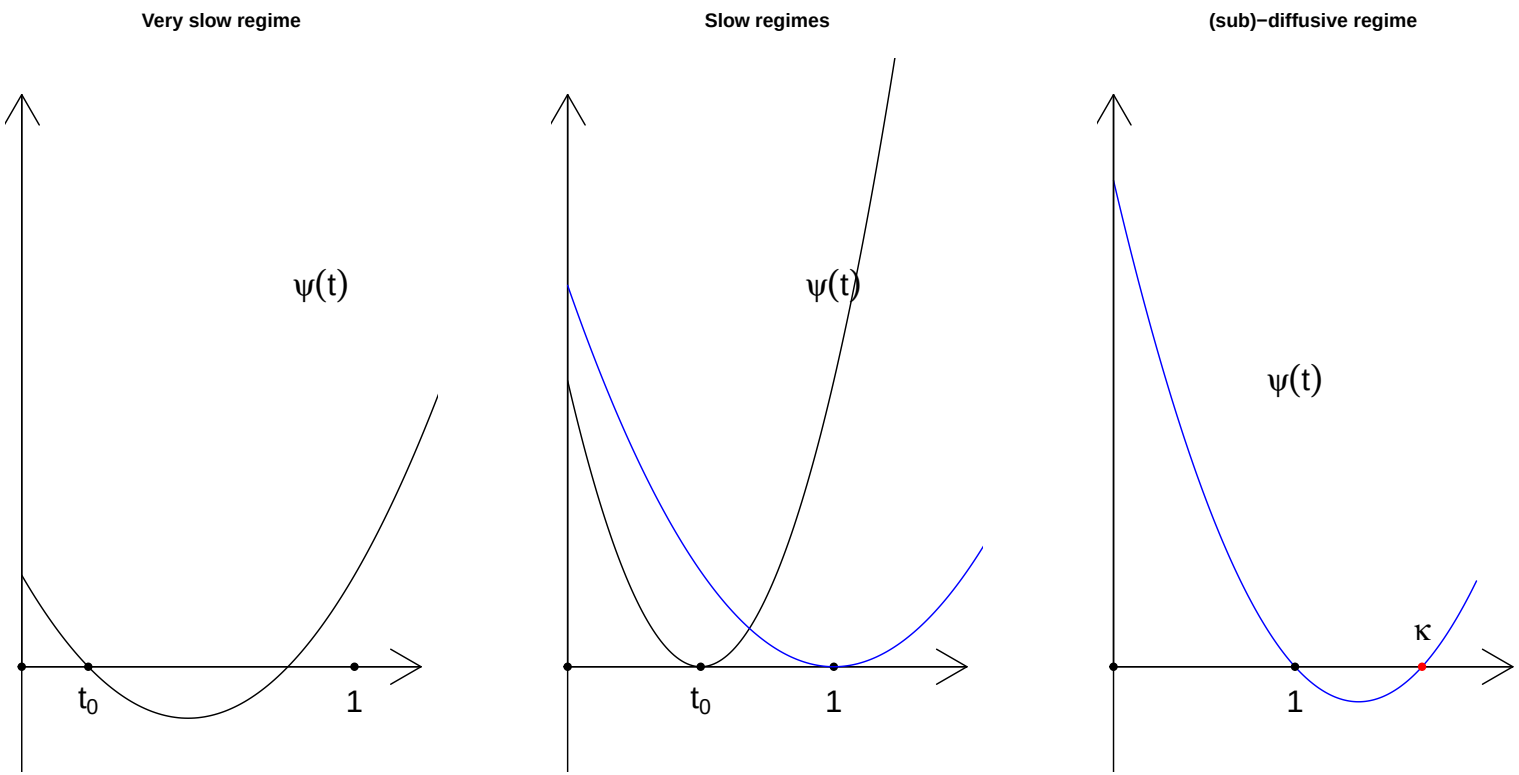

Figure: 4 regimes: Positive recurrent/Null recurrent

$$\underbrace{\max_{k\leq n}|X_k| = \Theta(\log n)}_{\text{very slow [Hu-Shi'07b]}}, \underbrace{\max_{k\leq n}|X_k| = \Theta((\log n)^3)}_{\text{slow [Hu-Shi'07a]}}, \underbrace{|X_n| = n^{1-\frac{1}{\kappa\wedge 2}+o(1)}}_{\text{(sub)-diffusive [HS'07b]}}$$

Null recurrent case: $\psi(1) = 0$, $\psi'(1) \leq 0$.

- Slow regime: $\psi(1) = \psi'(1) = 0$. [Hu-Shi'16] proved that under annealed probability $\mathbb{P}$,

$$\frac{|X_n|}{(\log n)^2} \text{ converges in law.}$$

- Sub-diffusive/diffusive regime: $\psi(1) = 0$, $\psi'(1) < 0$.
  1. $\kappa > 2$, [Aïdékon-de Raphélis'17]: under annealed and quenched law, $(\frac{|X_{\lfloor nt \rfloor}|}{\sqrt{n}})_{t \geq 0} \Rightarrow$ reflected Brownian motion.
  2. $\kappa = 2$, [de Raphélis'16]: under annealed and quenched law, $(\frac{\sqrt{\log n}}{\sqrt{n}} |X_{\lfloor nt \rfloor}|)_{t \geq 0} \Rightarrow$ reflected Brownian motion.
  3. $\kappa \in (1, 2)$, [de Raphélis'16]: under annealed and quenched law, $(\frac{1}{n^{1-\frac{1}{\kappa}}} |X_{\lfloor nt \rfloor}|)_{t \geq 0}$ converges in law.

Table of Contents

1 Biased random walk on G-W tree

2 Edge local time and heavy range in the recurrent case

3 Convergences of heavy range: ideas of proof

Edge local time in the recurrent case

- Edge local time for (u_{parent}, u) up to time n:

$$\overline{L}_n(u) := \sum_{k=1}^{n} \mathbf{1}_{\{(X_{k-1},X_k)=(u_{parent},u)\}}.$$

- Range up to time n: $R_n = \sum_{u\in\mathbb{T}} \mathbf{1}_{\{\overline{L}_n(u)\geq 1\}}$.

Range up to time n in null recurrent case

1. Slow regime: [Andreoletti-C.'18], in annealed probability,

$$\frac{\log n}{n} R_n \xrightarrow{\text{in } \mathbb{P}^*} \Lambda_0.$$

2. Sub-diffusive/diffusive regime: [de Raphélis'16], a.s.,

$$\frac{R_n}{n} \to C_0.$$

Λ_0 and C_0 are positive constants.

Heavy range of recurrent biased random walk

- Heavy range of level a up to time n [Andreoletti-Diel'18]:

$$R_n^{\geqslant a} := \sum_{u \in \mathbb{T}} \mathbf{1}_{\{\overline{L}_n(u) \geq a\}}.$$

Heavy range of recurrent biased random walk

- Heavy range of level a up to time n [Andreoletti-Diel'18]:

$$R_n^{\geqslant a} := \sum_{u \in \mathbb{T}} \mathbf{1}_{\{\overline{L}_n(u) \geq a\}}.$$

- The n-th return to ρ_{parent} or the n-th crossing from ρ_{parent} to ρ:

$$\forall n \geq 1,\ \tau_n := \inf\{k > \tau_{n-1} : (X_{k-1}, X_k) = (\rho_{parent}, \rho)\}.$$

with $\tau_0 = 0$. Then, $\tau_n < \infty$ a.s.

- Usually, we consider

$$R_{\tau_n}^{\geqslant n^\theta} := \sum_{u \in \mathbb{T}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}}.$$

- $(\overline{L}_{\tau_n}(u), u \in \mathbb{T})$ is a multi-type GW tree under annealed law $\mathbb{P}$, with types in $\mathbb{N}$. And, the quenched mean is $\mathbb{E}^{\mathcal{E}}[\overline{L}_{\tau_n}(u)] = ne^{-V(u)}$.

Known results under $\mathbb{P}^* = \mathbb{P}(\cdot|\mathbb{T} = \infty)$

Theorem [Andreoletti-Diel'18]

For $\forall \theta \geq 0$, $R_{\tau_n}^{\geqslant n^\theta} = n^{\xi_\theta + o(1)}$ in the annealed probability $\mathbb{P}(\cdot|\mathbb{T} = \infty)$, where $\xi_\theta \geq 0$ is a constant.

- Positive recurrent case: $\xi_\theta = t_0(1-\theta)_+$.
- Null recurrent case: $\xi_\theta = 0$ if $\theta \geq 1$; if $\theta \in [0,1)$,

$$\text{if } \Psi(1) = \Psi'(1) = 0, \xi_\theta = 1 - \theta.$$

$$\text{if } \Psi(1) = 0, \Psi'(1) < 0, \kappa = \inf\{t > 1, \Psi(t) = 0\} \in (1, \infty]$$

$$\xi_\theta = \begin{cases} \kappa(1-\theta) \text{ if } \kappa \in (1,2]; \\ \max(2 - \kappa\theta, 1-\theta) \text{ if } \kappa > 2. \end{cases}$$

Known results under $\mathbb{P}^* = \mathbb{P}(\cdot|\mathbb{T} = \infty)$

Theorem [Andreoletti-Diel'18]

For $\forall \theta \geq 0$, $R_n^{\geqslant n^\theta} = n^{\tilde{\xi}_\theta + o(1)}$ in the annealed probability $\mathbb{P}(\cdot|\mathbb{T} = \infty)$, where $\tilde{\xi}_\theta \geq 0$ is a constant.

- Positive recurrent case: $\tilde{\xi}_\theta = \xi_\theta = t_0(1-\theta)_+$ as $\tau_n =_{\mathbb{P}} \Theta(n)$.
- Null recurrent case:

$$\text{if } \Psi(1) = \Psi'(1) = 0, \tilde{\xi}_\theta = \xi_\theta = 1 - \theta, \text{ as } \tau_n =_{\mathbb{P}} \Theta(n \log n)$$

$$\text{if } \Psi(1) = 0, \Psi'(1) < 0, \kappa = \inf\{t > 1, \Psi(t) = 0\} \in (1, \infty]$$

$$\tilde{\xi}_\theta = \begin{cases} (1 - \kappa\theta)_+ \text{ if } \kappa \in (1, 2]; \\ (\max(1 - \kappa\theta, 1/2 - \theta))_+ \text{ if } \kappa > 2, \end{cases}$$

as $\tau_n =_{\mathbb{P}} n^{\kappa \wedge 2 + o(1)}$.

Main results

Theorem [C.'20+]

If $\Psi(1) = \Psi'(1) = 0$ (slow regime), under mild condition, in annealed probability $\mathbb{P}^*$,

$$\frac{R_{\tau_n}^{\geqslant n^\theta}}{n^{1-\theta}} \xrightarrow{n\to\infty} \Lambda(\theta) D_\infty,$$

with constant $\Lambda(\theta) \in (0,\infty)$ and $D_\infty \in (0,\infty)$ the a.s. limit of the derivative martingale $\sum_{|u|=n} V(u)e^{-V(u)}$.

Corollary

[Hu-Shi'16] showed that $\frac{\tau_n}{n\log n}$ converges in probability $\mathbb{P}^*$. So, $\frac{\log n}{n^{1-\theta}} R_n^{\geqslant n^\theta}$ converges in probability $\mathbb{P}^*$.

Main results

Theorem [C.-de Raphélis, in progress]

If $\Psi(1) = 0, \Psi'(1) < 0$ ((sub)-diffusive regime), under mild condition, for $\mathbf{P}^*$-a.s. environment, the following convergence holds under the quenched law $\mathbb{P}^{\mathcal{E}}$,

$$\left(\frac{\tau_n}{a_n}, \frac{R_{\tau_n}^{\geqslant n^\theta}}{k_n}\right) \xrightarrow{n\to\infty} (S_\kappa, \chi_\kappa).$$

where $k_n = n^{\xi_\theta}\mathbf{1}_{\{\kappa\neq 2\}} + \frac{n^{2(1-\theta)}}{\log n}\mathbf{1}_{\{\kappa=2\}}$,

$$a_n = n^\kappa \mathbf{1}_{\{\kappa\in(1,2)\}} + \frac{n^2}{\log n}\mathbf{1}_{\{\kappa=2\}} + n^2\mathbf{1}_{\{\kappa>2\}}.$$

Convergence up to time n

Let $\mathbf{ML}_n := \max_u \overline{L}_n(u)$.

Remark

We could then obtain that for $\mathbf{P}^*$-a.s. environment, under the quenched law,

- for $\kappa \in (1,2)$ and $\theta \in (0, \frac{1}{\kappa})$, $\frac{R_n^{\geqslant n^\theta}}{n^{1-\kappa\theta}}$ and $\frac{\mathbf{ML}_n}{n^{1/\kappa}}$ converge in law;
- for $\kappa = 2$ and $\theta \in (0, \frac{1}{2})$, $\frac{R_n^{\geqslant n^\theta}}{n^{1-2\theta}}$ and $\frac{\mathbf{ML}_n}{\sqrt{n \log n}}$ converge in law;
- for $\kappa > 2$ and $\theta \in (0, \frac{1}{2(\kappa-1)}]$, $\frac{R_n^{\geqslant n^\theta}}{n^{1-\kappa\theta}}$ and $\frac{\mathbf{ML}_n}{n^{1/2}}$ converge in law;
- for $\kappa > 2$ and $\theta \in (\frac{1}{2(\kappa-1)}, \frac{1}{2})$, $\frac{R_n^{\geqslant n^\theta}}{n^{\frac{1}{2}-\theta}}$ and $\frac{\mathbf{ML}_n}{n^{1/2}}$ converge in law.

Table of Contents

1 Biased random walk on G-W tree

2 Edge local time and heavy range in the recurrent case

3 Convergences of heavy range: ideas of proof

Ideas of proof: Multi-type GW tree from edge local times: $type(u) = \overline{L}_{\tau_n}(u), \forall u \in \mathbb{T}$

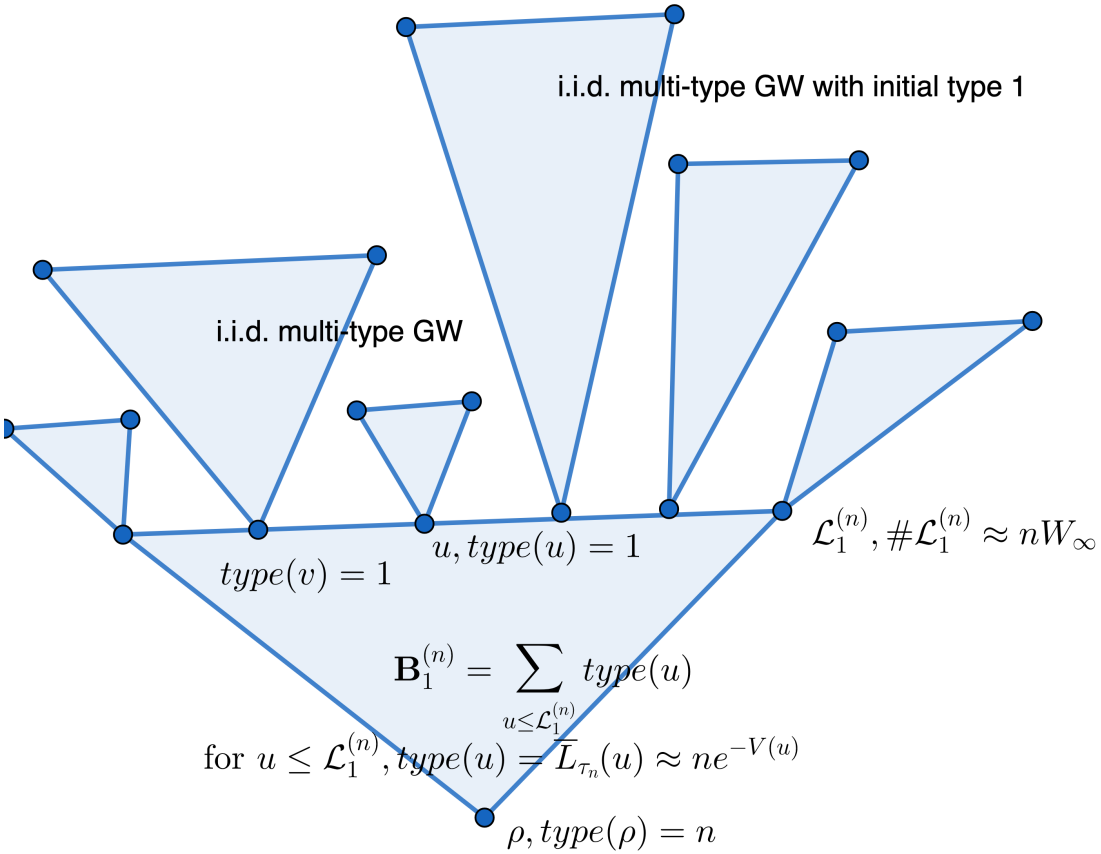

Figure: Stopping line:
$\mathcal{L}_1^{(n)} := \{u \in \mathbb{T} : \min_{\rho \le y < u} type(y) \ge 2, type(u) = 1\}$.

Ideas of proof: subdiffusive/diffusive regime

$$R_{\tau_n}^{\geqslant n^\theta} = \underbrace{\sum_{u \leq \mathcal{L}_1^{(n)}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}}}_{\Sigma_1} + \underbrace{\sum_{u > \mathcal{L}_1^{(n)}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}}}_{\Sigma_2}$$

Ideas of proof: subdiffusive/diffusive regime

$$R_{\tau_n}^{\geqslant n^\theta} = \underbrace{\sum_{u \leq \mathcal{L}_1^{(n)}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}}}_{\Sigma_1} + \underbrace{\sum_{u > \mathcal{L}_1^{(n)}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}}}_{\Sigma_2}$$

- For Σ_1, as $\mathbb{E}^{\mathcal{E}}[\overline{L}_{\tau_n}(u)] = ne^{-V(u)}$,

$$\Sigma_1 \approx \sum_{u \in \mathbb{T}} \mathbf{1}_{\{V(u) \leq (1-\theta)\log n\}} \sim C_1 n^{1-\theta} W_\infty,$$

where $W_\infty = \lim_{n\to\infty} \sum_{|u|=n} e^{-V(u)} \in (0,\infty)$ $\mathbf{P}^*$-a.s.

Ideas of proof: subdiffusive/diffusive regime

$$R_{\tau_n}^{\geqslant n^\theta} = \underbrace{\sum_{u \leq \mathcal{L}_1^{(n)}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}}}_{\Sigma_1} + \underbrace{\sum_{u > \mathcal{L}_1^{(n)}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}}}_{\Sigma_2}$$

- For Σ_1, as $\mathbb{E}^{\mathcal{E}}[\overline{L}_{\tau_n}(u)] = ne^{-V(u)}$,

$$\Sigma_1 \approx \sum_{u \in \mathbb{T}} \mathbf{1}_{\{V(u) \leq (1-\theta)\log n\}} \sim C_1 n^{1-\theta} W_\infty,$$

  where $W_\infty = \lim_{n\to\infty} \sum_{|u|=n} e^{-V(u)} \in (0,\infty)$ $\mathbf{P}^*$-a.s.

- For Σ_2, it is related to $\#\mathcal{L}_1^{(n)}$ i.i.d. multi-type GW tree with initial type 1 where under annealed probability, $\frac{\#\mathcal{L}_1^{(n)}}{n} \xrightarrow{\text{in } \mathbb{P}^*} W_\infty \in (0,\infty)$

I.i.d. multi-type GW trees with initial type 1

From multi-type GW tree (up to τ_1) to monotype GW tree $\mathbb{T}^1$.

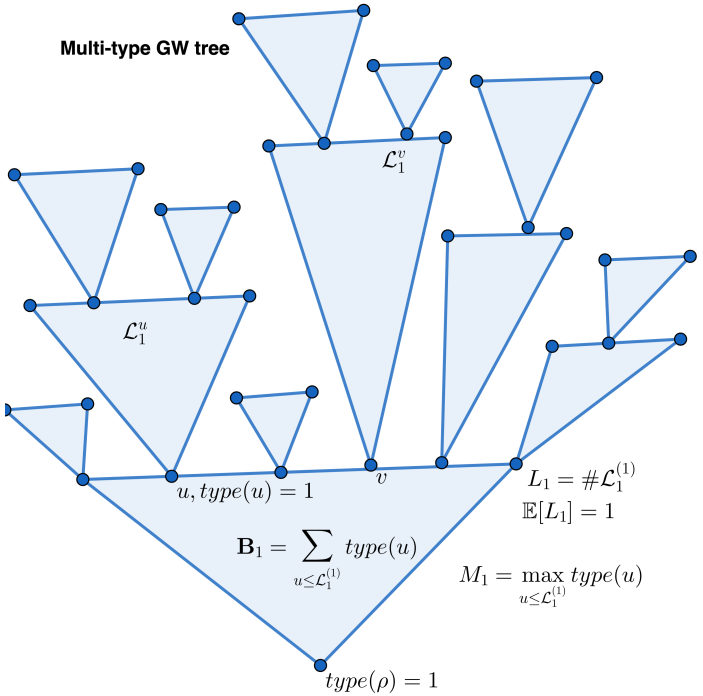

Figure: Multi-type GW tree

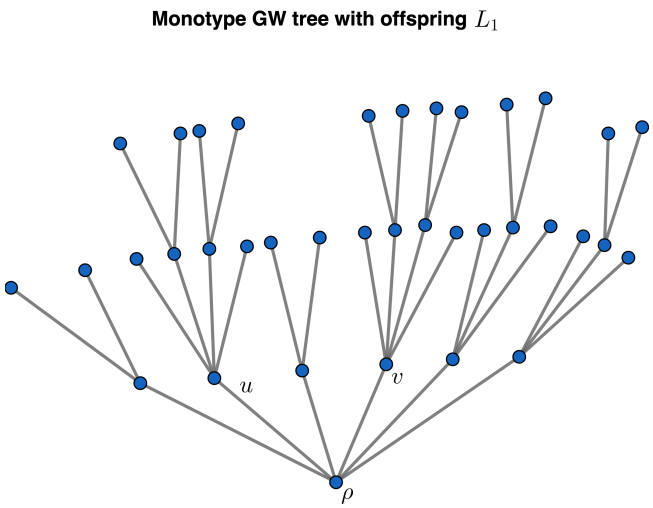

Figure: Monotype GW tree

$\tau_1 = 2\sum_{u\in\mathbb{T}^1} \mathbf{B}_1^u$ with $\mathbf{B}_1^u =_d \mathbf{B}_1$, $\sum_{u\in\mathbb{T}^1} 1 =$ total progeny.

Ideas of proof: sub-diffusive/diffusive regime

$\mathbb{T}^1 = \{u \in \mathbb{T} : type(u) = 1\} \to$ Monotype GW tree with offspring L_1.

- $\mathbb{E}[L_1] = 1$, $\mathbb{P}(L_1 \geq r) \sim c_L r^{-\kappa}$ see [de Raphélis'16].

Ideas of proof: sub-diffusive/diffusive regime

$\mathbb{T}^1 = \{u \in \mathbb{T} : type(u) = 1\} \to$ Monotype GW tree with offspring L_1.

- $\mathbb{E}[L_1] = 1$, $\mathbb{P}(L_1 \geq r) \sim c_L r^{-\kappa}$ see [de Raphélis'16].
- $\Sigma_2 = \sum_{i=1}^{\#\mathcal{L}_1^{(n)}} \sum_{u \in \mathbb{T}^1(i)} \sum_{u \leq v \leq \mathcal{L}_1^u} \mathbf{1}_{\{type(v) \geq n^\theta\}}$.
- $\sum_{u \leq v \leq \mathcal{L}_1^u} \mathbf{1}_{\{type(v) \geq n^\theta\}} \geq 1 \Leftrightarrow M_1^u \geq n^\theta$.

Ideas of proof: sub-diffusive/diffusive regime

$\mathbb{T}^1 = \{u \in \mathbb{T} : type(u) = 1\} \to$ Monotype GW tree with offspring L_1.

- $\mathbb{E}[L_1] = 1$, $\mathbb{P}(L_1 \geq r) \sim c_L r^{-\kappa}$ see [de Raphélis'16].
- $\Sigma_2 = \sum_{i=1}^{\#\mathcal{L}_1^{(n)}} \sum_{u \in \mathbb{T}^1(i)} \sum_{u \leq v \leq \mathcal{L}_1^u} \mathbf{1}_{\{type(v) \geq n^\theta\}}$.
- $\sum_{u \leq v \leq \mathcal{L}_1^u} \mathbf{1}_{\{type(v) \geq n^\theta\}} \geq 1 \Leftrightarrow M_1^u \geq n^\theta$.
- $\mathbb{P}(L_1 \geq ar, M_1 \geq r) \sim \gamma(a) r^{-\kappa}$ for any $a \geq 0$.
- $\mathbb{E}[\sum_{u \leq \mathcal{L}_1^{(1)}} \mathbf{1}_{\{type(u) \geq n^\theta\}} | M_1 \geq n^\theta] \xrightarrow{n \to \infty} C_0$.

Ideas of proof: sub-diffusive/diffusive regime

$\mathbb{T}^1 = \{u \in \mathbb{T} : type(u) = 1\} \to$ Monotype GW tree with offspring L_1.

- $\mathbb{E}[L_1] = 1$, $\mathbb{P}(L_1 \geq r) \sim c_L r^{-\kappa}$ see [de Raphélis'16].
- $\Sigma_2 = \sum_{i=1}^{\#\mathcal{L}_1^{(n)}} \sum_{u \in \mathbb{T}^1(i)} \sum_{u \leq v \leq \mathcal{L}_1^u} \mathbf{1}_{\{type(v) \geq n^\theta\}}$.
- $\sum_{u \leq v \leq \mathcal{L}_1^u} \mathbf{1}_{\{type(v) \geq n^\theta\}} \geq 1 \Leftrightarrow M_1^u \geq n^\theta$.
- $\mathbb{P}(L_1 \geq ar, M_1 \geq r) \sim \gamma(a) r^{-\kappa}$ for any $a \geq 0$.
- $\mathbb{E}[\sum_{u \leq \mathcal{L}_1^{(1)}} \mathbf{1}_{\{type(u) \geq n^\theta\}} | M_1 \geq n^\theta] \xrightarrow{n \to \infty} C_0$.

Ideas of proof: sub-diffusive/diffusive regime

- $$\Sigma_2 = \sum_{i=1}^{\#\mathcal{L}_1^{(n)}} \sum_{u\in\mathbb{T}^1(i)} \sum_{u\le v\le \mathcal{L}_1^u} \mathbf{1}_{\{type(v)\ge n^\theta\}}$$
$$\approx \sum_{i=1}^{nW_\infty} \sum_{u\in\mathbb{T}^1(i)} C_0 \underbrace{\mathbf{1}_{\{M_1^u\ge n^\theta\}}}_{L_1^u\ge \eta n^\theta} \to \text{large jumps of Lukasiewicz}$$
path of $\#\mathcal{L}_1^{(n)} \approx nW_\infty$ i.i.d. critical GW trees

Ideas of proof: sub-diffusive/diffusive regime

- $$\Sigma_2 = \sum_{i=1}^{\#\mathcal{L}_1^{(n)}} \sum_{u\in\mathbb{T}^1(i)} \sum_{u\le v\le \mathcal{L}_1^u} \mathbf{1}_{\{type(v)\ge n^\theta\}}$$
$$\approx \sum_{i=1}^{nW_\infty} \sum_{u\in\mathbb{T}^1(i)} C_0 \underbrace{\mathbf{1}_{\{M_1^u\ge n^\theta\}}}_{L_1^u\ge \eta n^\theta} \to \text{large jumps of Lukasiewicz}$$
$$\text{path of } \#\mathcal{L}_1^{(n)} \approx nW_\infty \text{ i.i.d. critical GW trees}$$

- So, $\frac{\Sigma_2}{b_n}$ converges in law with
$$b_n = n^{\kappa(1-\theta)}\mathbf{1}_{\{\kappa\in(1,2)\}} + \frac{n^{2(1-\theta)}}{\log n}\mathbf{1}_{\{\kappa=2\}} + n^{(2-\kappa\theta)}\mathbf{1}_{\{\kappa\in(2,\frac{2}{\theta})\}}.$$

Ideas of proof: slow movement regime $\psi(1) = \psi'(1) = 0$

- $\mathbb{E}[L_1] = 1 \to$ critical GW tree. Nevertheless, $\mathbb{P}(L_1 \geq r) \sim ?$

Ideas of proof: slow movement regime $\psi(1) = \psi'(1) = 0$

- $\mathbb{E}[L_1] = 1 \to$ critical GW tree. Nevertheless, $\mathbb{P}(L_1 \geq r) \sim ?$
- Nb of excursions visiting u:
  $E_n(u) := \sum_{k=1}^{n} \mathbf{1}_{\{\overline{L}_{\tau_k}(u) - \overline{L}_{\tau_{k-1}}(u) \geq 1\}}$.
- Under quenched $\mathbb{P}^{\mathcal{E}}$, $E_n(u) \sim Bin(n, p_u)$ where

$$p_u = \mathbb{P}^{\mathcal{E}}_{\rho}(\text{ hitting u before return to } \rho_{parent}) \approx e^{-\overline{V}(u)}$$

with $\overline{V}(u) = \max_{y:\text{ ancestor of } u} V(y)$.

Ideas of proof: slow movement regime $\psi(1) = \psi'(1) = 0$

- $\mathbb{E}[L_1] = 1 \to$ critical GW tree. Nevertheless, $\mathbb{P}(L_1 \geq r) \sim ?$
- Nb of excursions visiting u: $E_n(u) := \sum_{k=1}^{n} \mathbf{1}_{\{\overline{L}_{\tau_k}(u) - \overline{L}_{\tau_{k-1}}(u) \geq 1\}}$.
- Under quenched $\mathbb{P}^{\mathcal{E}}$, $E_n(u) \sim Bin(n, p_u)$ where

$$p_u = \mathbb{P}^{\mathcal{E}}_{\rho}(\text{ hitting u before return to } \rho_{parent}) \approx e^{-\overline{V}(u)}$$

  with $\overline{V}(u) = \max_{y:\text{ ancestor of } u} V(y)$.
- Let $\mathcal{E}_1^{(n)} := \{u \in \mathbb{T} : E_n(u) = 1, E_n(u_{parent}) \geq 2\}$ instead of $\mathcal{L}_1^{(n)}$. $\mathcal{E}_1^{(n)} \leq \mathcal{L}_1^{(n)}$.
- $R_{\tau_n}^{\geq n^\theta} = \sum_{u < \mathcal{E}_1^{(n)}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}} + \sum_{u \geq \mathcal{E}_1^{(n)}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u) \geq n^\theta\}}$

Ideas of proof: slow movement regime

- $$R_{\tau_n}^{\geqslant n^\theta} = \underbrace{\sum_{u\in\mathbb{T}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u)\geq n^\theta, E_n(u)\geq 2\}}}_{\Sigma_1} + \underbrace{\sum_{u\in\mathbb{T}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u)\geq n^\theta, E_n(u)=1\}}}_{\Sigma_2}$$
- For Σ_1:

$$\begin{aligned}\mathbf{1}_{\{\overline{L}_{\tau_n}(u)\geq n^\theta, E_n(u)\geq 2\}} &\approx \mathbf{1}_{\{ne^{-V(u)}\geq n^\theta, ne^{-\overline{V}(u)}\geq 2\}} \\ &\approx \mathbf{1}_{\{V(u)\leq(1-\theta)\log n, \overline{V}(u)\leq \log n\}}\end{aligned}$$

- So,
$\Sigma_1 \approx \sum_{u\in\mathbb{T}} \mathbf{1}_{\{V(u)\leq(1-\theta)\log n, \overline{V}(u)\leq\log n\}} \sim_{\mathbf{P}} C_1 D_\infty n^{1-\theta}$
where $D_\infty = \lim_{n\to\infty} \sum_{|u|=n} V(u)e^{-V(u)}$.

Ideas of proof: slow movement regime

- $R_{\tau_n}^{\geqslant n^\theta} = \underbrace{\sum_{u\in\mathbb{T}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u)\geq n^\theta, E_n(u)\geq 2\}}}_{\Sigma_1} + \underbrace{\sum_{u\in\mathbb{T}} \mathbf{1}_{\{\overline{L}_{\tau_n}(u)\geq n^\theta, E_n(u)=1\}}}_{\Sigma_2}$
- For Σ_1:

$$\begin{aligned}\mathbf{1}_{\{\overline{L}_{\tau_n}(u)\geq n^\theta, E_n(u)\geq 2\}} &\approx \mathbf{1}_{\{ne^{-V(u)}\geq n^\theta, ne^{-\overline{V}(u)}\geq 2\}}\\ &\approx \mathbf{1}_{\{V(u)\leq(1-\theta)\log n, \overline{V}(u)\leq \log n\}}\end{aligned}$$

- So,
  $\Sigma_1 \approx \sum_{u\in\mathbb{T}} \mathbf{1}_{\{V(u)\leq(1-\theta)\log n, \overline{V}(u)\leq\log n\}} \sim_{\mathbf{P}} C_1 D_\infty n^{1-\theta}$
  where $D_\infty = \lim_{n\to\infty} \sum_{|u|=n} V(u)e^{-V(u)}$.
- For Σ_2: $\Sigma_2 \approx \mathbb{E}^{\mathcal{E}}(\Sigma_2) \sim_{\mathbf{P}} C_2 D_\infty n^{1-\theta}$.

Further questions

Positive recurrent case:

- $\inf_{t\in[0,1]} \Psi(t) < 0$: very slow regime.

$$\mathbf{1}_{\{\overline{L}_{\tau_n}(u)\geq n^\theta\}} \approx \mathbf{1}_{\{ne^{-V(u)}\geq n^\theta\}} = \mathbf{1}_{\{V(u)\leq(1-\theta)\log n\}}.$$

- $\inf_{t\in[0,1]} \Psi(t) = \Psi(t_0) = 0$ and $\Psi'(1) > 0$: slow regime. It should be like the regime where $\Psi(1) = \Psi'(1) = 0$.

Question: tail of L_1 in the slow movement regime?
The work in progress with P. Andreoletti: $\mathbb{P}(\tau_1 \geq n) \sim \frac{\log n}{n}$.
Let $\mathcal{E}_1^{(n)} := \{u \in \mathbb{T} : E_n(u) = 1, E_n(u_{parent}) \geq 2\}$ instead of $\mathcal{L}_1^{(n)}$. $\mathcal{E}_1^{(n)} \leq \mathcal{L}_1^{(n)}$.

$$\#\mathcal{E}_1^{(n)} \approx_{\mathbb{P}^*} \frac{n}{\log n} D_\infty, \max_{u>\mathcal{E}_1^{(n)}} \overline{L}_{\tau_n}(u) =_{\mathbb{P}^*} \Theta(\frac{n}{\log n})$$