Lecture 3 Characteristic classes of 4-manifolds

Reference: Manolescu's notes 1.9–1.11

- Basic properties of char. classes

$\boxed{\text{Euler class}}$: let $E \xrightarrow{p} X$ be a real, oriented vector bundle of dimension n.

Q: when can we find a nowhere vanishing section $s: X \to E$?

A: Primary obstruction: $e(E) \in H^n(X;\mathbb{Z})$ called Euler class.

Idea: Construct s inductively on $X_k \leftarrow$ k-skeleton.

Suppose we have constructed $s|_{X_{k-1}}$, want to extend through k-cells $\{D^k_\ell\}_\ell$. I.e., we try to extend $s|_{\partial D^k_\ell}: \partial D^k_\ell = S^{k-1} \to \mathbb{R}^n - \{0\} \simeq S^{n-1}$ to D^k_ℓ. When $k<n$, this can always be done. So $s|_{X_{n-1}}$ always exists.

When $k=0$, this can be done iff $\deg(s|_{\partial D^k_\ell})=0$.

Facts: (1) The cochain $\tilde{e} \in C^n(X)$ $D^k_\ell \longmapsto \deg(s|_{\partial D^k_\ell})$ is closed.

(2) $e(E) := [\tilde{e}] \in H^n(X;\mathbb{Z})$ doesn't depend on $s|_{X_{n-1}}$.

(3) $p^{-1}(X_n) \xrightarrow{p} X_n$ has a nowhere vanishing section iff $e(E)=0$.

(4) Geometric definition of $e(E)$ when X is a manifold:

Take a generic section $s: X \to E$ which is transverse to zero section $s_0: X \to E$. Then $s^{-1}(0) := \{x \in X \mid s(x) = s_0(x)\}$ is a submanifold of X of codimension n. $e(E) = P.D.([s^{-1}(0)])$.

When X is oriented smooth manifold, define $\chi(X) := e(T_*X)[X] \in \mathbb{Z}$

Theorem (Poincaré-Hopf) $\chi(X) = \sum_{i=0}^{\dim X} (-1)^i b_i(X)$

Other properties:

- orientation $e(\bar{E}) = -e(E)$
- functorality $f^*(e(E)) = e(f^*(E))$ $\quad E \xrightarrow{p} X$, $f: Y \to X$
- product $e(E \oplus F) = e(E) \cup e(F)$

$\boxed{\text{Chern class}}$ Take $\mathbb{C}^n \hookrightarrow E \to X$.

Q: Can we find sections $s_1, s_2, \cdots, s_k : X \to E$ which are linearly independent everywhere?

$V_k(\mathbb{C}^n) = \{\text{linearly indep } (v_1, \cdots, v_k) \in \mathbb{C}^n \times \cdots \times \mathbb{C}^n\}$

Thm (Steenrod) $V_k(\mathbb{C}^n)$ is $2(n-k)$-connected. $\pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) \cong \mathbb{Z}$.

The primary obstruction $c_{n-k+1}(E) \in H^{2(n-k+1)}(X; \mathbb{Z})$

Chern classes are uniquely characterized by the following axioms:

- rank $c_0(E) = 1$, $c_k(E) = 0 \ \forall k > \dim(E)$
- naturality $f: Y \to X$ $\quad f^*(c_k(E)) = c_k(f^*(E))$
- product Let $c(E) = c_0(E) + \cdots + c_{\dim E}(E)$ ← total chern class

  Then $c(E \oplus F) = c(E) \cup c(F)$

- normalization $c(T_*\mathbb{CP}^n) = (1+\omega)^{n+1}$ where $\omega = P.D[\mathbb{CP}^{n-1}]$

Note: all char. classes for complex bundles are some combination of chern classes.

Stiefel-Whitney class

Let $\mathbb{R}^n \hookrightarrow E \xrightarrow{P} X$ be a real vector bundle (orientable or not).

Q: When can we find k sections $s_1, \cdots, s_k : X \to \mathbb{R}^n$ that are linearly independent?

A: If $(n+1-k)$ is even, then primary obstruction is a class $w_{n+1-k}(E) \in H^{n+1-k}(X; \mathbb{Z}/2)$

If $(n+1-k)$ is odd, then primary obstruction is a class $\tilde{w}_{n+1-k}(E) \in H^{n+1-k}(X; \tilde{\mathbb{Z}})$ where $\tilde{\mathbb{Z}}$ is a local system over X

Let's do a reduction $\tilde{\mathbb{Z}} \rightsquigarrow \mathbb{Z}/2$ and get $w_{n+1-k}(E)$

$w_*(E)$ is called the Stiefel-Whitney class. It's characterized by functoriality, product, rank $\quad w_0(E) = 1$, $w_k(E) = 0$ if $k > \dim E$

normalization $w_*(T_*\mathbb{RP}^n) = (1+w)^{n+1}$, $w = P.D.[\mathbb{RP}^{n-1}] \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$

It's the universal char. class in $H^*(-; \mathbb{Z}/2)$ for real bundles.

Pontryagin classes

$P_i(E) := (-1)^i C_{2i}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(X; \mathbb{Z})$, $\mathbb{R}^n \hookrightarrow E \to X$

Note: the odd chern classes of $E \otimes_{\mathbb{R}} \mathbb{C}$ are all 2-torsion.

Pontrajin class satisfies the functorality / natural property and $P_*(E \oplus F) - P_*(E) \cup P_*(F)$ is 2-torsion.

Additional properties

- $c_{\dim(E)}(E) = e(E)$ for complex E
- $w_{\dim(E)}(E) \equiv e(E) \bmod 2$ for real E
- E is a complex bundle $\Rightarrow$ $w_{2i+1}(E) = 0$, $w_{2i}(E) = c_i(E) \bmod 2$.
- For real E $w_1(E) = 0 \Leftrightarrow E$ is orientable

  If $w_1(E) = 0$, then E is spinable $\Leftrightarrow w_2(E) = 0$

Spinable: Consider the frame bundle $SO(n) \hookrightarrow Fr \to X$ defined by

$Fr_x := \{$ordered, orientation compatible orthonormal basis of $E_x\}$

A spin structure is a lift of Fr to $Spin(n) \hookrightarrow \widetilde{Fr} \to X$.

We say E is spinable if E has some spin structure.

(Spin structure is usually not unique. If we fix one spin structure S_0 then we have bi-jection $H^1(X; \mathbb{Z}/2) \xleftrightarrow{1:1} \{\text{spin str. on } E\}$)

$$\alpha \longmapsto S_0 \text{ twisted by } \alpha$$

Characteristic class of 4-manifolds

X: smooth 4-mfd, oriented.

- $e(T_*X)[X] = \sum (-1)^i b_i(X)$
- $P_0(T_*X) = 1$, $P_k(T_*(X)) = 0 \ \forall k \geq 2$

Theorem (Hirzebruch) $P_1(TX)[X] = 3\sigma(X)$

- $w_0(TX) = 1$ $w_1(TX) = 0$ $w_4(TX) = e(TX) \bmod 2$

Theorem (Wu) (1) $\forall \alpha \in H^2(X;\mathbb{Z}/2)$ we have $w_2(TX) \cup \alpha = \alpha \cup \alpha$

(I.e. $w_2(TX)$ is charactistic). This uniquely determines $w_2(TX)$.

(2) $w_3(TX) = Sq^1(w_2(TX))$.

Let's prove a weaker version using geometric method (can be generalized.)

($Q_X(\alpha,\beta) \overset{\text{written as}}{\rightsquigarrow} \langle \alpha, \beta \rangle$)

Lemma: $\forall \alpha \in H^2(X;\mathbb{Z})$ $\langle w_2(TX), \alpha \rangle = \langle \alpha, \alpha \rangle \mod 2$.

Proof: $\alpha = P.D.[\Sigma]$ for some $\Sigma \hookrightarrow X$

$$
\begin{aligned}
\langle w_2(TX), \alpha \rangle &= w_2(TX) \cdot [\Sigma] \\
&= w_2(TX|_\Sigma) \cdot [\Sigma] \\
&= w_2(T\Sigma \oplus N\Sigma)\, [\Sigma] \\
&= (w_2(T\Sigma) + w_2(N\Sigma)) \cdot [\Sigma] \\
&= e(T\Sigma) \cdot [\Sigma] + e(N\Sigma)[\Sigma] \mod 2 \\
&= \chi(\Sigma) + \Sigma \cdot \Sigma \mod 2 \\
&= \langle \alpha, \alpha \rangle
\end{aligned}
$$

$\square$

Corollary: X is spin $\Leftrightarrow w_2(TX) = 0 \Rightarrow Q_X$ is even

("$\Leftarrow$" if $\pi_1(X) = 1$)

Recall: For smooth. X with $\pi_1(X) = 1$, we have

$b_2(X), \sigma(X)$, type of Q_X (odd/even) $\xrightarrow{\text{Donadson + classification of bilinear forms}} Q_X$

$\xrightarrow{\text{Freedman}}$ homeomorphism type of X.

Corollary: Let X be a smooth, closed 4-mfd. Then homeomorphism type of X is determined by $e(TX)$, $p_1(TX)$ and $w_2(TX)$ (zero/nonzeoro)

Characteristic classes in dimension 4 can't detect exotic phenomina.

In higher dimensions, $e(TX)$ and $w_*(TX)$ only depend on homotopy type.

Theorem (Novikov) $p_i(TX)_{\mathbb{Q}} \in H^{4i}(X;\mathbb{Q})$ only depends on homeomorphism type of X.

Note that we know examples where $p_i(TX)$ is not topological invariant.

Algebraic surfaces in $\mathbb{C}P^3$

P: homogeneous polynomial with 4 variables, degree $d>0$.

$X_p := \{[(z_0,\cdots,z_3)] \in \mathbb{C}P^3 \mid p(z_0,\cdots,z_3)=0\}$

We say X_p is smooth (as algebraic surface) if the equations

$\left\{\frac{\partial P}{\partial z_i}=0\right\}_{0\le i\le 3} \cup \{p(z_0,\cdots,z_3)=0\}$ have no nonzero solutions.

$$\{\text{non-zero poly of degree } d\}/\text{Scalar} = \mathbb{C}P^{f(d)} \leftarrow \text{some function}$$

$$\cup$$

$$\{p \mid X_p \text{ is smooth}\} \leftarrow \text{Zariski open, connected.}$$

So diffeomorphism type of X_p only depends on $d=\deg(P)$.

This allows us to write X_d if we only care about smooth str.

In particular, we can take $p(z_0,\cdots,z_3)=\sum z_i^d$

Example: $X_1 \cong_{diff} \mathbb{C}P^2$ (See e.g. Miles Reid algebraic surfaces)

$X_2 \cong_{diff} S^2 \times S^2$

$X_3 \cong_{diff} \mathbb{C}P^2 \# 6\overline{\mathbb{C}P}^2$

$X_4 \cong_{diff} K3$

X_5 is a surface of general type. We have

$X_5 \cong_{top} 9\mathbb{C}P^2 \# 44\overline{\mathbb{C}P}^2$ but $X_5 \not\cong_{diff} 9\mathbb{C}P^2 \# 44\overline{\mathbb{C}P}^2$

$\boxed{\text{Rest of today: compute } H_*(X_d)}$

Theorem (Lefschetz hyperplane theorem) Let X be a hypersurface in $\mathbb{C}P^n$ (defined by a single polynomial). Then $X \hookrightarrow \mathbb{C}P^n$ induces iso. on $\pi_k(-)$ if $k < n-1$.

Corollary: $\pi_1(X_d) = 0$. $\forall d > 0$.

Next: $c_1(T_* X_d)$, $c_2(T_* X_d)$. $X = X_d$

H: tautological line bundle over $\mathbb{C}P^3$

$$H = (\mathbb{C}^4 - \{0\}) \times \mathbb{C} / (\vec{v}, z) \sim (\omega\vec{v}, \omega z) \quad \omega \in \mathbb{C} - \{0\}$$

$c_1(H) = \eta \in H^2(\mathbb{C}P^3; \mathbb{Z})$ Then P.D. $[X_d] = d \cdot \eta$

$\eta|_X := \eta_X$

Any degree d-polynomial $p: \mathbb{C}^4 \to \mathbb{C}$ induces a section

$$S_p: \mathbb{C}P^3 \to \underbrace{H \otimes \cdots \otimes H}^{d}$$

0-loci of $S_p = X$. $\Rightarrow$ $NX = H^d|_X$

$$(1+\eta_X)^4 = (1+\eta)^4|_X = c(T\mathbb{C}P^3|_X) = c(TX) \cup c(NX)$$
$$= c(TX) \cdot c(1 + d\eta_X)$$

$\Rightarrow c_1(TX) = (4-d)\eta_X$ $\quad c_2(TX) = (6 + d^2 - 4d)\eta_X^2$

$c_1(TX) = (4-d)\eta_X \qquad c_2(TX) = (6+d^2-4d)\eta_X^2$

$\eta_X^2[X] = (P.D.[X] \cup \eta^2)[\mathbb{CP}^3] = d$

So we see
$$\left.\begin{aligned} c_2(TX)[X] &= (6+d^2-4d)\,d \\ (c_1(TX))^2[X] &= (4-d)^2\cdot d \end{aligned}\right\} \text{Chern numbers}$$

Note: $2+b_2(X) = \chi(X) = e(TX)[X] = (6+d^2-4d)\cdot d$

Definition: An almost complex structure J on a smooth mfd Y is a smooth collection $\{J: T_yY \to T_yY\}_{y\in Y}$ s.t. $J_y^2 = -1 \ \forall y$.

(complex structure $\Rightarrow$ almost complex st.r., "$\Leftarrow$" if Nijenhuis tensor $= 0$)

Given J, we can regard T_*Y as a complex vector bundle.

Lemma: For any almost complex manifold Y^4, we have
$$c_1^2(TY)[Y] = 3\sigma(Y) + 2\chi(Y)$$

Proof:
$$\begin{aligned} 3\sigma(Y) &= p_1(TY)[Y] \\ &= -c_2(TY\otimes_{\mathbb{R}}\mathbb{C})[Y] \\ &= -c_2(TY\oplus\overline{TY})[Y] \\ &= -\left(c_2(TY) + c_2(\overline{TY}) + c_1(TY)\cdot c_1(\overline{TY})\right)[Y] \\ &= \left(c_1(TY)^2 - 2c_2(TY)\right)[Y] \\ &= c_1(TY)^2 - 2\chi(Y) \end{aligned}$$
$\square$

so $(4-d)^2 \cdot d = 3\sigma(X) - 2(2+b_2(X))$

$(6+d^2-4d)\cdot d = 2+b_2(X)$

$\Rightarrow \sigma(X) = \dfrac{d(4-d^2)}{3}$, $\quad b_2(X) = d^3-4d^2+6d-2$

Finally, since $w_2(TX) = c_1(TX) \bmod 2$

$= d\eta_X \bmod 2$

We see Q_X is even $\Leftrightarrow w_2(TX)=0 \Leftrightarrow d$ is even

$\Updownarrow$

X is spin.

So we know rank, signature, type of $Q_X \xrightarrow{\text{determines}} Q_X$

Special cases:

$d=4$ $\quad b_2(X)=22$ $\quad \sigma(X)=-16$ $\quad Q_X$ is even.

By classification theorem of bi-linear form $Q_X = 2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$d=5$ $\quad b_2(X)=53$ $\quad \sigma(X)=-35$ $\quad Q_X$ is odd

$\Rightarrow Q_X = 9\langle 1\rangle \oplus 44\langle -1\rangle \xRightarrow{\text{Freedman}} X \cong_{top} (9\mathbb{CP}^2)\#(44\overline{\mathbb{CP}^2})$

Two results from Seiberg-Witten theory $\Rightarrow X \not\cong_{diff} (9\mathbb{CP}^2)\#(44\overline{\mathbb{CP}^2})$

Theorem (Taubes) Let X be a symplectic 4-mfd. Then

$X \not\cong_{diff} X_1 \# X_2$ with $b_2^+(X_1), b_2^+(X_2) > 0$.

Theorem (Witten, Taubes) Let X be a symplectic 4-mfd with $b_2^+(X)>0$. Then X doesn't have a positive scalar curvature metric.