2021-11-10 Kähler geometry

Remark Because of $\mathbb{C}^*$-action ①

$$(M_t, \mathcal{L}|_{M_t}) \cong (M_s, \mathcal{L}|_{M_s})$$

for all $t \neq 0$ and $s \neq 0$.

Remark Flatness implies that for r large

$\underbrace{\pi_* \mathcal{L}}$ $\longrightarrow \mathbb{C}$ is a vector bundle.

(sheaf of $H^0(U, \mathcal{L}|_U)$, $U \subset \mathbb{C}$)

of rank $h^0(M, L^r) := \dim H^0(M, L^r)$

$$\begin{array}{ccl} \mathcal{M} & \longrightarrow & \mathbb{P}^{h_0 - 1}(\mathbb{C}) \quad \mathbb{C}^*\text{-equiv.} \\ \cup & \nearrow & \\ M_t & & \text{embedding} \end{array}$$

Def (product configuration).

Suppose (M, L) admits a $\mathbb{C}^*$-action.

Then $(M \times \mathbb{C}, L^r \times \mathbb{C})$ is naturally a test configuration.

This is called the product configuration.

Remark Given (M, L), there are many test configurations.

Definition of Donaldson - Futaki invariant. 2

Let (M_0, L_0) be a polarized scheme.

$\dim M_0 = m$.

Let $w(k)$ be the weight of the $\mathbb{C}^*$-action on $H^0(M_0, L_0^k)$.

Computed by equivariant Riemann-Roch (Berline - Getzler - Vergne), polynomial in k of degree $m+1$

weight: $\bigwedge^{h_0(L^k)} H^0(M_0, L_0^k) \cong \mathbb{C}$

$z \mapsto t^{-w(L)} z$

Let $d(k) = \dim H^0(M_0, L_0^k)$.

Computed by Riemann-Roch, polynomial in k of degree m.

$$\underset{\substack{\parallel \\ h^0(L^k)}}{\chi} = \frac{c_1(L^k)^m}{m!} + \text{lower order in } k.$$

by Kodaira vanishing.

$$\frac{w(k)}{k\, d(k)} = F_0 + F_1 k^{-1} + \cdots$$

Def $DF(\mathcal{M}) = -F_1$ for (M_0, L_0)

"Donaldson - Futaki invariant."

Def A polarized manifold (M, L) is said to be K-polystable if for any test configuration $(\mathcal{M}, \mathcal{L})$, the central fiber $(M_0, \mathcal{L}|_{M_0})$ has non-positive F_1 $(DF(\mathcal{M}) \geq 0)$, and $F_1 = 0$ if and only if $(\mathcal{M}, \mathcal{L})$ is a product configuration.

Yau-Tian-Donaldson conjecture

Let (M, L) be a polarized manifold. There exists a Kähler metric of constant scalar curvature in the Kähler class $c_1(L)$ if and only if (M, L) is K-polystable.

Remark For (M, K_M^{-1}) Fano, this was solved by Chen-Donaldson-Sun, Tian 2012 ~ 2015.

Remark This conj may not be true.

For updated conj, see

- S. K. Donaldson arXiv. 1808.03995
- R. Dervan - J. Ross Math. Res. Letters vol 24 (2017).

Remark For Toric surfaces, OK by Donaldson.

For toric case OK for any dimension.

Chen - Cheng ;

Weiyong He :) $\to$ Legendre

Lemma When (M_0, L_0) is smooth.

$$F_1 = -\text{const}\ f(X) \left(= \text{const} \int_M u_X S \omega^n \right)$$

const > 0.

where X is the infinitesimal generator of the $\mathbb{C}^*$-action.

Proof $w(k)$ is computed by the equivariant Riemann-Roch formula

$w(k)$ = the degree 1 Term in t of

$$\int_M e^{k(\omega + t u_X)} \, Td(t\, L(X) + \Theta)$$

where $i(X)\omega = -\sqrt{-1}\,\bar\partial u_X$, $\int u_X \omega^m = 0$,

θ type $(1,0)$-connection of $T'M$

$\Theta = d\theta + \theta\wedge\theta$ curvature form.

$\Theta^{0,2} = 0$.

$$L(X) = \nabla_X - L_X \in C^\infty(M, \mathrm{End}\, T'M).$$

See Berline - Getzler - Vergne.'s book.

$d(k)$ is given by R. Roch.

Todd

$$d(k) = \int_M e^{k\omega}\, Td(M).$$

Write $w(k)$ and $d(k)$

$$w(k) = b_0 k^{m+1} + b_1 k^m + \cdots$$

$$d(k) = a_0 k^m + a_1 k^{m-1} + \cdots$$

Then

$$a_0 = \frac{1}{m!}\int_M \omega^m = \frac{1}{m!}\int c_1(L)^m = vol(M, \omega).$$

$$a_1 = \frac{1}{(m-1)!}\int_M \overbrace{c_1(L)^{k-1}}^{\omega^{m-1}} \wedge \frac{1}{2}\overbrace{c_1(T'M)}^{\text{Ricci}}$$

$$= \frac{1}{2(m-1)!}\int \rho_\omega \wedge \omega^{m-1} = \frac{1}{2\, m!}\int_M \underset{\text{Scal}(\omega)}{S}\, \omega^m$$

$$\left(S = g^{i\bar{j}} R_{i\bar{j}} \right)$$

$$b_0 = \frac{1}{(m+1)!} \underbrace{\int_M (m+1)\, u_X\, \omega^m}_{} \longrightarrow 0$$

normalization

$$b_1 = \frac{1}{m!} \int_M m\, u_X\, \omega^{m-1} \wedge \frac{1}{2} c_1(M)$$

$$+ \underbrace{\frac{1}{m!} \int \underline{\operatorname{div} X} \cdot \omega^m}_{} \longrightarrow 0$$

$$\left(\begin{aligned} (\nabla_X - L_X)(Y) &= \nabla_X Y - [X, Y] = \nabla_Y X \\ \nabla_X - L_X &= \nabla X \end{aligned} \right)$$

divergence thm.

$$\frac{w(k)}{k\, d(k)} = \frac{b_0 k^{m+1} + b_1 k^m + \cdots}{a_0 k^{m+1} + a_0 k^m + \cdots}$$

$$= \frac{b_0}{a_0} \frac{k^{m+1} + \frac{b_1}{b_0} k^m + \cdots}{k^{m+1} + \frac{a_1}{a_0} k^m + \cdots}$$

$$= \frac{b_0}{a_0} \left(1 + \left(\frac{b_1}{b_0} - \frac{a_1}{a_0} \right) k^{-1} + \cdots \right)$$

$$\therefore\ F_1 = \frac{b_0}{a_0} \left(\frac{b_1}{b_0} - \frac{a_1}{a_0} \right) = \frac{1}{a_0^2} (a_0 b_1 - a_1 b_0)$$

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$$F_1 = \frac{1}{\mathrm{vol}(M,\omega)^2}\left(\frac{\mathrm{vol}((M)}{2}\int_M u_X S \frac{\omega^m}{m!}\right)$$

$$= \frac{1}{2\mathrm{vol}(M,\omega)}\int_M u_X S \frac{\omega^m}{m!}$$

$$F_1 = -\frac{1}{2m!\,\mathrm{vol}(M,\omega)} f(X)$$

:-)

My original def

$$S - S_0 = \Delta F$$

$$f(X) = \int XF\omega^n = \int u^i F_i \,\omega^n$$

$$= -\int u\,\Delta F\,\omega^n$$

$$= -\int u\,(S - S_0)\,\omega^m.$$

$$= -\int u S\,\omega^n$$

$$\int u\,\omega^m = 0$$

Work of Xiaowei Wang:
Moment maps, Futaki invariant and stability of projective manifolds, Comm. Anal. Geom. 12 (2004), 187-196.

Lichnerowicz-Matsushima, Futaki can be both interpreted as obstructions to stability in terms of finite dimensional moment map picture. ($\exists$ finite dimensional analogues.)

$$\begin{array}{ccl} \Lambda & \longrightarrow & N \quad \text{ample line bundle. f. n. dim} \\ & & \downarrow \\ \mathcal{L} & \longrightarrow & Z \quad \text{infinite di.} \end{array}$$

$$\bigwedge^{\dim} \ker P_p \otimes \left(\bigwedge^{\dim} \operatorname{Im} P_p\right)^*$$

Tian (1997)

$\exists \mathcal{L} \to$ space of Kähler forms as the determinant line bundle of some family of elliptic operators such that log(Quillen metric) = Mabuchi K-energy

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" analytic Torsion "
W. Müller - K. Wendland

" CM-line bundle "

Question : Properness of K-energy.

Tian : The behavior of K-energy at infinity is approximated by " Futaki invariant ".

Need to the degeneration of M.
normal varieties Ding-Tian defined generalized Futaki invariant
" special degeneration " (1997)

Alg geometers thought " normality condition " is a drawback because degeneration of alg variety is not normal in general.

2002. Donaldson defined " test configuration "
" Donaldson-Futaki inv " without " normality ".