Cohomology and sheaves

We'll come back to this in more detail later. For now, we just explain enough to continue our discussion of mirror symmetry.

<u>Def</u> (partial) a <u>sheaf</u> of abelian groups $\mathcal{F}$ on a topological space M is given by (1) for each open $U \subset M$, an abelian group $\mathcal{F}(U)$ and (2) for opens $U \subset V$ a morphism $\mathcal{F}(V) \to \mathcal{F}(U)$ satisfying certain compatibilities.

<u>Ex</u> (1) For a vector bundle E on M, get a sheaf where $\mathcal{F}(U)$ is sections of $E|_U$, and morphisms are restrictions.

(2) For an abelian group G, get a sheaf $\underline{G}$ on M where $\mathcal{F}(U)$ is locally constant functions $U \to G$.

Complexes of sheaves

We can also consider morphisms between sheaves, and kernels and images of such. Then for sheaves

$$\cdots \longrightarrow \mathcal{F}^i \xrightarrow[d^i]{} \mathcal{F}^{i+1} \xrightarrow[d^{i+1}]{} \mathcal{F}^{i+2} \longrightarrow \cdots$$

say have <u>complex</u> if $d^{i+1} \circ d^i = 0$, and <u>exact complex</u> if furthermore have equality in $\ker(d^{i+1}) \supseteq \mathrm{im}(d^i)$. Failure of exactness is measured by $\mathcal{H}^i(\mathcal{F}^\bullet) = \ker(d^i)/\mathrm{im}(d^{i-1})$, <u>cohomology sheaf</u>.

<u>Ex</u> A sheaf of abelian groups on $M = pt$ is determined by a single abelian group $F = \mathcal{F}(pt)$. Morphism between such sheaves correspond to morphisms between abelian groups, and thence cohomology sheaves to cohomology groups.

Cohomology theories

Let's briefly review, for more details see [H, Appendix B]

Def a resolution $\mathcal{F}^\bullet$ of a sheaf $\mathcal{G}$ is a sequence $\mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$

such that $0 \to \mathcal{G} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$ is exact complex of sheaves

Čech cohomology groups

Take $M = \bigcup_{i=1-n} U_i$ open cover. Then have Čech complex $C^\bullet(\mathcal{F})$ of

abelian groups $C^j(\mathcal{F}) = \bigoplus_{i_0 < \cdots < i_j} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_j})$ with differentials

$d^j : C^j \to C^{j+1}$ defined using restriction morphisms. For instance,

$d^0 : \bigoplus_i F(U_i) \to \bigoplus_{a<b} F(U_a \cap U_b)$ with components res if $i = a$

and $-$res if $i = b$

so that $\ker d^0 = F(M)$. Then

Def Čech cohomology group $\check{H}^j(\mathcal{F}) = H^j(C^\bullet(\mathcal{F}))$

<u>Ex</u> $\check{H}^0(\mathcal{F}) = \mathcal{F}(M)$ "global sections of $\mathcal{F}$"

$\check{H}^j(\mathcal{F})$ for $j>0$ gives obstructions to existence of global sections.

<u>deRham cohomology groups</u>

Write Ω^j for the sheaf of j-forms on a smooth manifold M, with coefficients in $\mathbb{R}$. In particular, Ω^0 is sheaf of (smooth) functions.

Consider sequence $F^\cdot(M)$: $0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots$

<u>Def</u> deRham cohomology group $H^j_{dR}(M,\mathbb{R}) = H^j(F^\cdot(M))$

<u>Ex</u> $H^0_{dR}(M,\mathbb{R}) = \{$locally constant $\mathbb{R}$-valued functions on $M\}$

(solutions of equation $df = 0$)

<u>Prop</u> $H^j_{dR}(M,\mathbb{R}) = \check{H}^j(\underline{\mathbb{R}})$ locally constant sheaf on M

when $\{U_i\}$ is <u>good</u>, ie. all $U_{i_0} \cap \ldots \cap U_{i_j}$ are contractible or empty.

Proof Uses that $\mathcal{F}^\bullet : 0 \to \Omega^0 \to \Omega^1 \to \dots$ is a resolution of $\mathbb{R}$.

Sheaf cohomology

For a sheaf $\mathcal{F}$ on M we make:

Def (preliminary) $H^j(\mathcal{F}) = \check{H}^j(\mathcal{F})$ for a good cover.

Rem Later, we'll see that sheaf cohomology can be defined more elegantly in terms of a resolution of $\mathcal{F}$.

Complex manifolds

We may similarly use $\mathbb{C}$-valued functions to define $H^j_{dR}(X, \mathbb{C})$ on complex manifold X. Now a r-form with $\mathbb{C}$-coefficients may be written locally as a sum of (p,q)-forms $f\, dz_1 \wedge \dots \wedge dz_p \wedge d\bar{w}_1 \wedge \dots \wedge d\bar{w}_q$, $p+q=r$, for f a function and z_j, ω_j holomorphic functions (informally a decomposition into holomorphic and anti-holomorphic parts)

For a compact Kähler manifold, this has the following consequence on the level of cohomology.

$$\boxed{H^r_{dR}(X, \mathbb{C}) = \bigoplus H^{p,q}(X) \text{ where } H^{p,q}(X) = H^q(\Omega^p)}$$

Rem This follows from "Hodge theory" [H, §3.2]

We call $H^{p,q}(X)$ the Hodge groups, and refer to Hodge numbers $h^{p,q}(X) = \dim H^{p,q}(X)$

Symmetries of Hodge groups

We have $H^{p,q}(X) \cong H^{q,p}(X)$ via complex conjugation

Also Serre duality [H, §4.1] gives an isomorphism

$$H^{p,q}(X) \cong H^{n-p,n-q}(X)^\vee \quad \text{where } n = \dim X$$

It follows that

$$h^{p,q} = h^{q,p} \text{ and } h^{p,q} = h^{n-p,n-q}$$

Hodge diamond

It is convenient to arrange the Hodge numbers as follows:

$$\begin{array}{ccccc} & & h^{n,n} & & \\ & \cdot & \cdot & \cdot & \\ h^{n,0} & \cdot & \cdot & \cdot & h^{0,n} \\ & \cdot & \cdot & \cdot & \\ & & h^{0,0} & & \end{array}$$

Ex Smooth complex curve X, genus g.

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

$$H^{1,0}(X) = H^0(\Omega^1) = \Omega^1(X) = \mathbb{C}^g$$

$$H^{0,0}(X) = \mathcal{O}(X) = \mathbb{C}$$

Ex Abelian surface X, say $X = E \times F$ for E, F elliptic curves

$\uparrow$

Calabi–Yau 1-folds

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 2 & & 2 & \\
1 & & 4 & & 1 \\
 & 2 & & 2 & \\
 & & 1 & &
\end{array}
$$

Ex K3 surface X

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

$$\Omega^2_X = \omega_X \cong \mathcal{O}$$
$$\Rightarrow H^{2,0} = \mathbb{C}$$
$$\Rightarrow h^{2,0} = 1$$

To exclude the case of abelian surfaces it is common to take

Def A (strong) Calabi–Yau X is Calabi–Yau as above with furthermore $H^{p,0}$ trivial for $0 < p < n$

Rem K3 surfaces are Calabi-Yau in this sense

Now, for a strong Calabi-Yau 3-fold X we have

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

mirror

Here, by the above, have

$$h^{2,1} = h^{1,2} \text{ and } h^{1,1} = h^{2,2}$$

For a mirror partner X', mirror symmetry predicts

that $h^{1,1}(X) = h^{2,1}(X')$

and $h^{2,1}(X) = h^{1,1}(X')$

Idea: mirror symmetry reflects the Hodge diamond in the line shown.

Rem This is origin of term "mirror symmetry"

We outline how this comes from physics.