

Cohomology and sheaves

We'll come back to this in more detail later. For now, we just explain enough to continue our discussion of mirror symmetry.

Def (partial) a sheaf of abelian groups \mathcal{F} on a topological space M is given by ① for each open $U \subset M$, an abelian group $\mathcal{F}(U)$ and ② for opens $U \subset V$ a morphism $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$ satisfying certain compatibilities.

Ex ① For a vector bundle E on M , get a sheaf where $\mathcal{F}(U)$ is sections of $E|_U$, and morphisms are restrictions.

② For an abelian group G , get a sheaf \underline{G} on M where $\mathcal{F}(U)$ is locally constant functions $U \rightarrow G$.

Complexes of sheaves

We can also consider morphisms between sheaves, and kernels and images of such. Then for sheaves

$$\dots \rightarrow \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \mathcal{F}^{i+2} \rightarrow \dots$$

say have complex if $d^{i+1} \circ d^i = 0$, and exact complex if furthermore

have equality in $\ker(d^{i+1}) \supseteq \operatorname{im}(d^i)$. Failure of exactness is

measured by $H^i(\mathcal{F}^\bullet) = \ker(d^i) / \operatorname{im}(d^{i-1})$, cohomology sheaf.

Ex A sheaf of abelian groups on $M = \text{pt}$ is determined by

a single abelian group $F = \mathcal{F}(\text{pt})$. Morphism between

such sheaves correspond to morphisms between abelian

groups, and thence cohomology sheaves to cohomology groups.

Cohomology theories

Let's briefly review, for more details see [H, Appendix B]

Def a resolution \mathcal{F}^\bullet of a sheaf \mathcal{G} is a sequence $\mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots$

such that $0 \rightarrow \mathcal{G} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots$ is exact complex of sheaves

Čech cohomology groups

Take $M = \bigcup_{i=1}^n U_i$ open cover. Then have Čech complex $C^\bullet(\mathcal{F})$ of

abelian groups $C^j(\mathcal{F}) = \bigoplus_{i_0 < \dots < i_j} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_j})$ with differentials

$d^j : C^j \rightarrow C^{j+1}$ defined using restriction morphisms. For instance,

$$d^0 : \bigoplus_i \mathcal{F}(U_i) \rightarrow \bigoplus_{a < b} \mathcal{F}(U_a \cap U_b) \text{ with components } \text{res if } i=a \\ \text{and } -\text{res if } i=b$$

so that $\ker d^0 = \mathcal{F}(M)$. Then

Def Čech cohomology group $\check{H}^j(\mathcal{F}) = H^j(C^\bullet(\mathcal{F}))$

Ex $\check{H}^0(\mathcal{F}) = \mathcal{F}(M)$ "global sections of \mathcal{F} "

$\check{H}^j(\mathcal{F})$ for $j > 0$ gives obstructions to existence of global sections.

deRham cohomology groups

Write Ω^j for the sheaf of j -forms on a smooth manifold M , with coefficients in \mathbb{R} . In particular, Ω^0 is sheaf of (smooth) functions.

Consider sequence $F^\bullet(M) : 0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots$

Def deRham cohomology group $H_{\text{dR}}^j(M, \mathbb{R}) = H^j(F^\bullet(M))$

Ex $H_{\text{dR}}^0(M, \mathbb{R}) = \{\text{locally constant } \mathbb{R}\text{-valued functions on } M\}$

(solutions of equation $df = 0$)

Prop $H_{\text{dR}}^j(M, \mathbb{R}) = \check{H}^j(\underline{\mathbb{R}})$ locally constant sheaf on M
when $\{U_i\}$ is good, i.e. all $U_{i_0} \cap \dots \cap U_{i_j}$
are contractible or empty.

Proof Uses that $\mathcal{F}^\bullet : 0 \rightarrow \mathcal{N}^0 \rightarrow \mathcal{N}^1 \rightarrow \dots$ is a resolution of \mathbb{R} .

Sheaf cohomology

For a sheaf \mathcal{F} on M we make:

Def (preliminary) $H^j(\mathcal{F}) = \check{H}^j(\mathcal{F})$ for a good cover.

Rem Later, we'll see that sheaf cohomology can be defined more elegantly in terms of a resolution of \mathcal{F} .

Complex manifolds

We may similarly use \mathbb{C} -valued functions to define

$H_{\text{dR}}^j(X, \mathbb{C})$ on complex manifold X . Now a r -form with \mathbb{C} -coefficients may be written locally as a sum

of (p, q) -forms $f dz_1 \wedge \dots \wedge dz_p \wedge d\bar{w}_1 \wedge \dots \wedge d\bar{w}_q$, $p+q=r$,

for f a function and z_j, ω_j holomorphic functions
(informally a decomposition into holomorphic
and anti-holomorphic parts.)

For a compact Kähler manifold, this has the
following consequence on the level of cohomology.

$$\boxed{H_{\text{dR}}^r(X, \mathbb{C}) = \bigoplus H^{p,q}(X) \text{ where } H^{p,q}(X) = H^q(\mathbb{Q}^p)}$$

Rem This follows from "Hodge theory" [H, §3.2]

We call $H^{p,q}(X)$ the Hodge groups, and refer to
Hodge numbers $h^{p,q}(X) = \dim H^{p,q}(X)$

Symmetries of Hodge groups

We have $H^{p,q}(X) \cong H^{q,p}(X)$ via complex conjugation

Also Serre duality [H, §4.1] gives an isomorphism

$$H^{p,q}(X) \cong H^{n-p, n-q}(X)^{\vee} \quad \text{where } n = \dim X$$

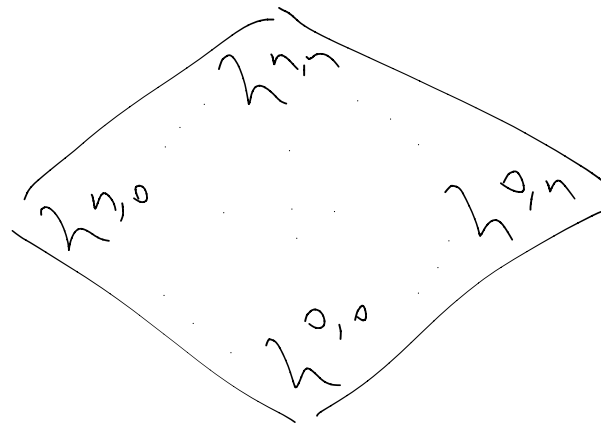
It follows that

$$h^{p,q} = h^{q,p} \quad \text{and} \quad h^{p,q} = h^{n-p, n-q}$$

Hodge diamond

It is convenient to arrange the Hodge numbers

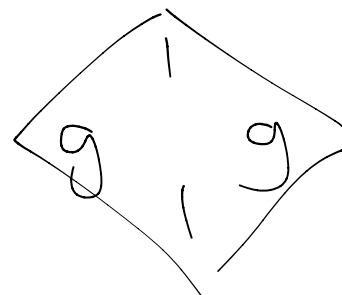
as follows:



Ex Smooth complex curve X , genus g .

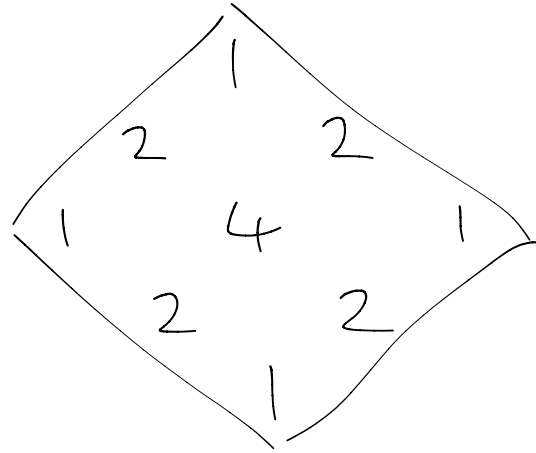
$$H^{1,0}(X) = H^0(\mathcal{O}_X') = \mathcal{O}_X'(X) = \mathbb{C}g$$

$$H^{0,0}(X) = \mathcal{O}(X) = \mathbb{C}$$



Ex Abelian surface X , say $X = E \times F$ for E, F elliptic curves

↑
Calabi-Yau 1-folds

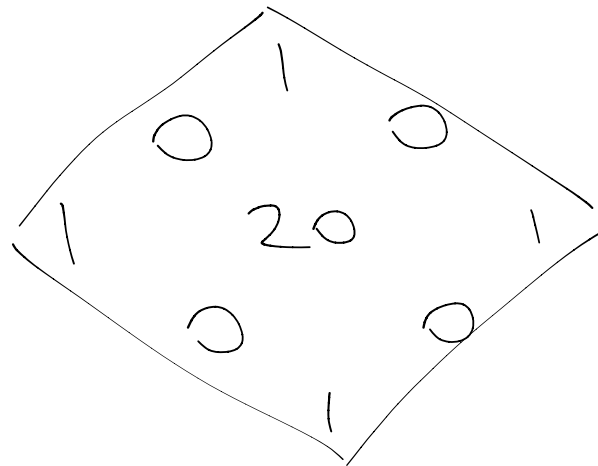


Ex K3 surface X

$$H^2_X = \omega_X \cong \mathbb{C}$$

$$\Rightarrow H^{2,0} = \mathbb{C}$$

$$\Rightarrow h^{2,0} = 1$$



To exclude the case of abelian surfaces it is common to take

Def A (strong) Calabi-Yau X is Calabi-Yau as above

with furthermore $H^{p,0}$ trivial for $0 < p < n$

Rem K3 surfaces are Calabi-Yau in this sense

Now, for a strong Calabi-Yau 3-fold X we have

Here, by the above, have

$$h^{2,1} = h^{1,2} \text{ and } h^{1,1} = h^{2,2}$$

For a mirror partner X' ,
mirror symmetry predicts

$$\text{that } h^{1,1}(X) = h^{2,1}(X')$$

$$\text{and } h^{2,1}(X) = h^{1,1}(X')$$

Idea: mirror symmetry reflects the Hodge diamond
in the line shown.

Rem This is origin of term "mirror symmetry"

We outline how this comes from physics.

