Lecture 13. Virtual knots

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Virtual links I

A virtual knot represents a natural combinatorial generalisation of a classical knot: We introduce a new type of a crossing and extend new moves to the list of the Reidemeister moves. The new crossing type (which is called virtual and depicted by a small circle) should be treated neither as a passage of one branch over the other one nor as a passage of one branch under the other. It should be treated as a diagrammatic picture of two parts of a knot (a link) on the plane which are far from each other, and the intersection of these parts is an artifact of such a drawing, see Fig. 1.

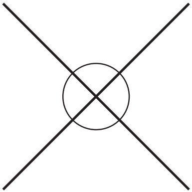

Figure 1: A virtual crossing

In that sense the following list of generalised Reidemeister moves is natural: All classical Reidemeister moves related to classical crossings and a detour move. The latter represents the following: A branch of a knot diagram containing several consecutive virtual crossings but not containing classical crossings can be transformed into any other branch with the same endpoints; new intersections and selfintersections are marked as virtual crossings, see Fig. 2.

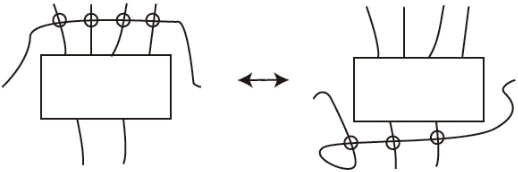

Figure 2: The detour moves

Definition 1.1

A virtual diagram (or a diagram of a virtual link) is the image of an immersion of a framed 4-valent graph in $\mathbb{R}^2$ with a finite number of intersections of edges. Moreover, each intersection is a transverse double point which we call a virtual crossing and mark by a small circle, and each vertex of the graph is endowed with the classical crossing structure (with a choice for underpass and overpass specified). The vertices of the graph are called classical crossings or just crossings.

A virtual diagram is called connected if the corresponding framed 4-valent graph is connected.

Definition 1.2

A virtual link is an equivalence class of virtual diagrams modulo the generalised Reidemeister moves. The latter consist of the usual Reidemeister moves referring to classical crossings and the detour move.

Thereby, in order to define a virtual knot, we need only to know the position of classical crossings and their connections with each other. Moreover, positions of paths connecting classical crossings, their intersections and self-intersections, are not important for us.
We also allow circles without vertices as separate connected components.

Virtual links II

Like a classical link a virtual link has the number of components. A virtual knot is a virtual link with one (unicursal) component. The components of a virtual link can be described combinatorially by using virtual diagrams.

Definition 1.3

By a unicursal component of a virtual diagram K we mean the following. Consider K as a one-dimensional cell-complex on the plane. Some of the connected components of this complex are circles. We call each of these components a unicursal component of K. The remaining part of the cell-complex represents a framed 4-valent graph with vertices which are classical or virtual crossings. Unicursal components of K are (besides circles) equivalence classes on the set of edges of the graph: Two edges e, e' are equivalent if there exists a collection of edges $e = e_1, \ldots, e_k = e'$ and a collection of vertices $v_1, \ldots v_{k-1}$ (some of them may coincide) of the graph such that edges e_i, e_{i+1} are opposite to each other at the vertex v_i.

It is easy to see that the number of components of a virtual diagram is invariant under the generalised Reidemeister moves. Therefore, we can define unicursal components of a link as unicursal components of a diagram of the link. In the classical case, this definition coincides with the definition given above.

Definition 1.4

The writhe, w(K), of a virtual diagram K is the number equal to the number of positive crossings  minus the number of negative crossings .

Definition 1.5

By the Gauss diagram corresponding to a planar diagram of a (virtual) knot we mean a diagram consisting of an oriented circle (with a fixed point, not a preimage of a crossing) on which the preimages of the overcrossing and the undercrossing (for each classical crossing) are connected by an arrow directed from the preimage of the undercrossing to the preimage of the overcrossing. Each arrow is endowed with a sign coinciding with the sign of a crossing, i.e. it equals 1 if we have  and -1 for .

The Gauss diagram can be viewed as a chord diagram with an additional structure [9].

Definition 1.6

A chord is odd (even) if it intersects an odd (even) number of other chords.

Formally, parity is a way of associating 0 and 1 to crossings which satisfies certain axioms. We have already seen Gaussian parity. There are many other parities, which we'll see in the next lectures.

Remark 1.7

Parity is a powerful tool to generate new invariants.

The Gauss diagram of the right-handed trefoil is shown in Fig. 3.

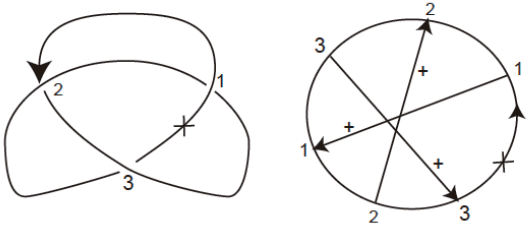

Figure 3: The Gauss diagram of the right-handed trefoil

Gauss diagrams (with some information lost; we forget about the orientation of chords and signs) can be encoded by double occurrence words, i.e. each letter in a word occurs precisely twice, and, moreover, it does not matter which letters are used, but it is important whether two letters are the same or not.

Let us attribute a letter to each classical crossing (different crossings have different letters). To each classical crossing, an arrow of the Gauss diagram is assigned. Let us place each letter corresponding to a crossing near the head and the tail of the arrow corresponding to the crossing. By traveling along the circle of the Gauss diagram from the fixed point and writing down consequently the letters met by us, we get the double occurrence word.

Let us call this word the Gauss code of a knot diagram. For instance, let us attribute the letters a, b, c to the crossings of the classical trefoil. Then the Gauss code is abcabc.

Arbitrary Gauss diagrams, generally speaking, can be non-realisable by an embedding of a framed 4-valent graph in the plane, but they can be realised by means of a generic immersion in the plane by making points having more than one preimage (in a generic case they have exactly two preimages) virtual crossings, see Fig. 4.

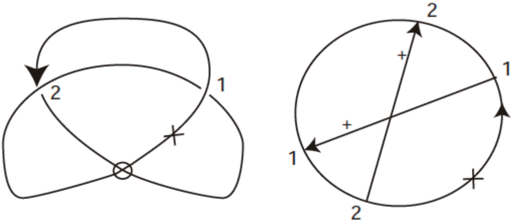

Figure 4: The Gauss diagram of a virtual knot

This naturally leads us to the following definition of a virtual knot (not links): One has to consider all formal Gauss diagrams and describe formal Reidemeister moves on them (as it was done in the case of classical knots). They will represent combinatorial scheme of transformations of Gauss diagrams. In that case, equivalence classes of Gauss diagrams over formal Reidemeister moves are virtual knots. Note that we do not need the detour move as a Gauss diagram "knows" nothing about position of virtual crossings on the plane, and it "knows" only position of classical crossings and their connections with each other. It means that Gauss diagrams "feel" only classical Reidemeister moves and do not "feel" the detour move.

In order to define a link, one can consider the generalisation of a Gauss diagram with several core circles. See Fig. 5

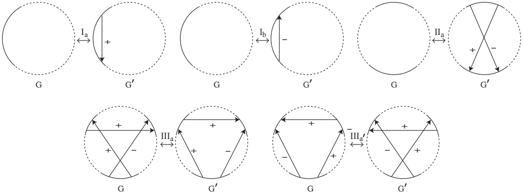

Figure 5: Reidemeister moves for Gauss diagrams of a virtual knot

Virtual knots (here by a knot we mean also a link) can be defined as knots in thickened oriented surfaces $S \times I$, where S is a two-dimensional oriented closed surface and $I = [0, 1]$ is the segment with a fixed orientation; moreover, thickened surfaces should be considered up to stabilisations, i.e. up to additions and removals of handles to S in such a way that additional thickened handles do not touch our knot. Branches of a virtual knot having a virtual intersection for a virtual diagram and related to two parts of the virtual knot located far away from each other, can move freely on the surface independently from each other.

From now on, we suppose that the structure of direct product is fixed on a thickened surface $S \times I$ and it is pointed toward which side is up, say the side $S \times \{1\}$, and which one is down, say the side $S \times \{0\}$. In the case of links, one can consider a disjoint union of manifolds, $S = S_1 \sqcup \cdots \sqcup S_k$.
Links in $S \times I$ can be described by diagrams on S with the under/ overcrossing structure specified. In that sense, virtual diagrams are obtained by means of regular generic projections of diagrams from S to the plane: Crossings pass to classical crossings and new intersections (artifacts) are marked by virtual crossings; moreover, it is required that under regular generic projections neighborhoods of classical crossings are mapped into the oriented plane with the orientation preserved. The Reidemeister moves for diagrams on S (the same as in the case of classical diagrams) correspond to the classical Reidemeister moves for virtual diagrams; there are also transformations which do not change the combinatorial structure of diagrams on S, but do change the combinatorial structure of the projection to the plane. The detour move corresponds to these transformations.

A realisation of the detour move by moves on thickened surfaces and their projections is shown in Fig. 6.

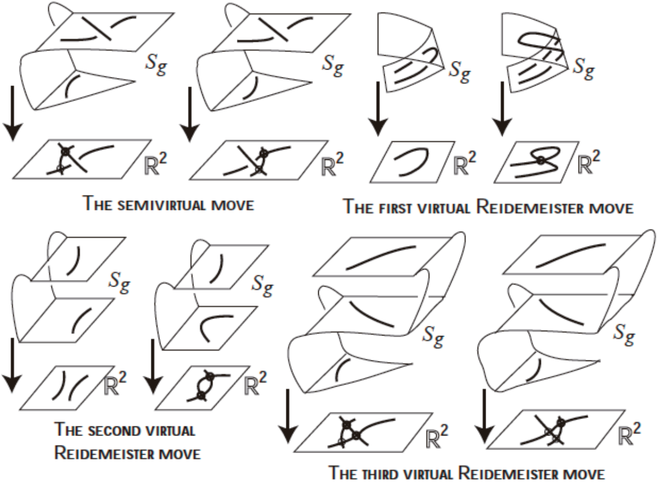

Figure 6: Generalised Reidemeister moves and thickened surfaces

This leads us to local versions of the detour move which consist of: (1) virtual Reidemeister moves $\Omega_1',\Omega_2',\Omega_3'$: which are obtained from the classical Reidemeister moves by swapping all classical crossings participating in moves for virtual crossings, see Fig. 7.

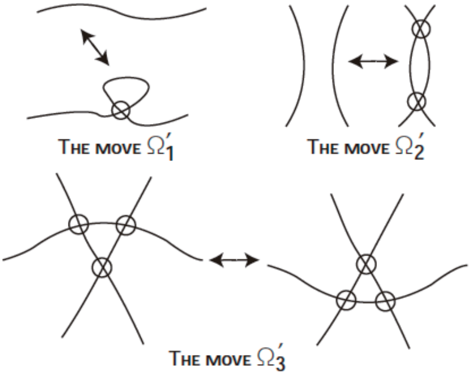

Figure 7: Moves $\Omega_1',\Omega_2',\Omega_3'$

(2) semivirtual version Ω_3'' of the third Reidemeister move. Under this move the branch containing two virtual crossings can be carried through a classical crossing, see Fig. 8.

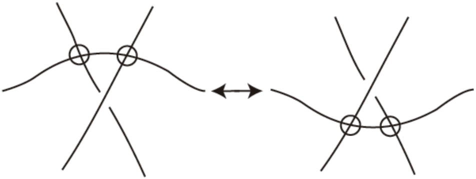

Figure 8: The semivirtual move Ω_3''

We call a Reidemeister move increasing (respectively, decreasing) if this move increases (respectively, decreases) the number of crossings (the number of classical crossings in the classical case and the number of virtual ones in the virtual case).

Proposition 1.8

Two virtual diagrams K and K′ are obtained from each other by a finite sequence of detour moves if and only if they are obtained from each other by a finite sequence of the moves Ω_1', Ω_2', Ω_3', Ω_3'' and their inverses.

The reconstruction of a knot diagram in a thickened surface

The reconstruction of a knot diagram in a thickened surface by a virtual diagram on the plane is as follows.

Let K be a virtual diagram on the sphere S^2 (we compactify the plane $\mathbb{R}^2$ by adding one point). Each virtual crossing of this diagram corresponds to an intersection of two arcs. Let us choose one of them and construct a handle for its "lifting", see Fig. 9.

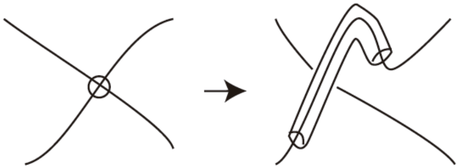

Figure 9: Lifting of a virtual crossing to a handle

As a result, we get a diagram (with over/undercrossings and virtual crossings) on the torus, the number of virtual crossings of which is less by one than the number of virtual crossings in the initial diagram. Note that the choice for a position of a handle, up or down, is immaterial since thickened surfaces are considered on their own account without any embedding into $\mathbb{R}^3$.

It is also easy to check that it does not matter which arc we choose for lifting it to a new handle. Two diagrams K_1 and K_2 corresponding to two such lifts to surfaces M_1 and M_2 with handles, i.e. $K_1 \subset M_1$ and $K_2 \subset M_2$, will be combinatorially equivalent (i.e. there exists a homeomorphism $f : M_1 \to M_2$ of one lifting to the other one and f transforms one virtual diagram to the other one, $f(K_1) = K_2$).

Diagrams on a sphere without virtual crossings give the same as diagrams of the plane (diagrams of classical knots). So, knots of genus 0 are just classical knots.

Continuing this process we can get rid of all the virtual crossings and obtain a diagram on $S_g(= S_g \times \{\frac{1}{2}\} \subset S_g \times [0,1])$, where g is the number of handles. Here it is convenient to use detour moves. Each of these moves is a merging of subsequently situated handles to one handle and a partition of this handle into new handles situated in other places, see Fig. 10.

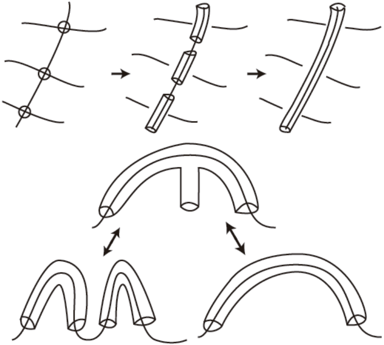

Figure 10: Detour move and stabilisation

In Fig. 10 (lower part) merging (respectively, partition) consists of elementary moves which are the destabilisation (respectively, the stabilisation). Meanwhile classical Reidemeister moves are performed locally on some part of the surface S_g obtained from the sphere by adding handles.

It is natural that the surface S_g is automatically oriented. The orientation for S_g arises from the orientation of the sphere S_g to which we attach handles.

Note that on the surface S_g there is no fixed system of longitudes and meridians. Actually, under the first virtual Reidemeister move this surface goes through Dehn twist, see Fig. 11.

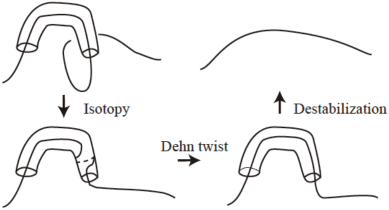

Figure 11: Dehn twist and the move Ω_1'

Theorem 1.9 ([6])

Every stable equivalence class of links in thickened surfaces has a unique irreducible representative.

Here, “irreducible” means “minimal genus”.

Remark 1.10

Recall that a surface with 0 genus is a sphere.

Odd writhe

Definition 1.11

Let K be a diagram of an oriented virtual knot. Call a classical crossing v of K odd, if in the Gauss code of K the number of letters between two occurrences of the crossing v is odd.

Set:

$$J(K) = w(K)|_{Odd(K)},$$

where Odd(K) denotes the collection of odd crossings of K, and the restriction of the writhe w(K) to Odd(K), denoted by $w(K)|_{Odd(K)}$, means the summation the signs of the odd crossings in K.

Then it is not hard to see that J(K) is an invariant of the virtual knot (link) K. We call J(K) the self-linking number of the virtual diagram K. This invariant is simple, but remarkably powerful.

If K is classical, then $J(K) = 0$, since there are no odd crossings at a classical diagram.

Theorem 1.12

Let K be a virtual knot diagram and let K^* denote the mirror image of K (obtained by switching all the crossings of the diagram K). Then

$$J(K*) = -J(K).$$

Hence, if $J(K) \neq 0$, then K is inequivalent to its mirror image. If K is a virtual knot and $J(K)$ is non-zero, then K is not equivalent to any classical knot.

Example 1.13

In the case of the virtual trefoil K two crossings are odd and, hence, we have $J(K) = 2$. This proves that K is non-trivial, non-classical and inequivalent to its mirror image. Similarly, for the virtual knot E the crossings a and b are odd. We have $J(E) = 2$ and, hence, the knot E is also non-trivial, non-classical and inequivalent to its mirror image. Note that for the knot E the invariant is independent of the type of the crossing c.

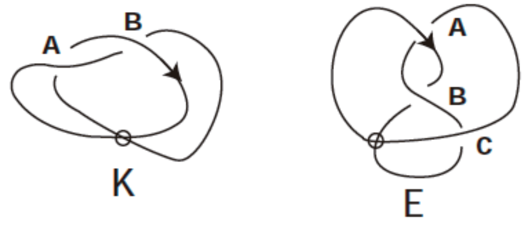

Figure 12: Virtual trefoil K and virtual figure eight E

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Fundamental groups

In previous section we have seen the phenomenon of parity which appears in virtual knot theory. There will be a lecture about free knots where we shall see that knot theories with parity (e.g., virtual knot theory) allows one to construct invariants of an absolutely new sort: picture-valued invariants. These invariants are valued not in groups or polynomials but in linear combinations of knot diagrams [8, 10].

Though virtual knots are not embeddings in $\mathbb{R}^3$, one can easily construct a generalisation of the knot complement fundamental group (or, simply, the knot group) for virtual knots. Namely, one can modify the Wirtinger presentation for virtual diagrams. Consider a diagram $\overline{\mathrm{L}}$ of a virtual link L. Instead of arcs we shall consider long arcs of $\overline{\mathrm{L}}$, which pass from an underpass to the next underpass. We take these arcs as generators of the group to be constructed. After this, we shall write down the relations at classical crossings just as in the classical case: if two long arcs a and c are divided by a long arc b, whence a lies on the right hand with respect to the orientation of b, then we write down the relation $c = bab^{-1}$.

Fundamental groups

The invariance of this group under classical Reidemeister moves can be checked straightforwardly: the combinatorial proof of this fact works both for virtual and classical knots. For the semivirtual move and purely virtual moves there is nothing to prove: we shall get the same presentation.

However, this invariance results from a stronger result: invariance of the virtual knot quandle, which will be discussed later.

Definition 2.1

The group defined as above is called the group of the link L.

Strange properties of virtual knots

Some virtual links may have properties that do not occur in the classical case. For instance, both in the classical and the virtual case one can define “upper” and “lower” presentations of the knot group (the first is as above, the second is just the same for the knot (or link) where all classical types are switched). In the classical case, these two presentations give the same group (for geometric reasons). In the virtual case it is however not so. The example first given in [1] is as follows.

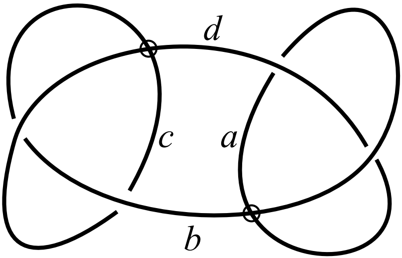

Figure 13: A virtual knot with different upper and lower groups

Strange properties of virtual knots

In fact, taking the arcs a, b, c, d shown in Fig. 13 as generators, we obtain the following relations:

$$b = dad^{-1}, \quad a = bdb^{-1}, \quad d = bcb^{-1}, \quad c = dbd^{-1}.$$

Thus, a and c can be expressed in the terms of b and d. So, we obtain the presentation $\langle b, d | bdb = dbd \rangle$. So, this group is isomorphic to the trefoil group.

Strange properties of virtual knots

Definition 2.2

By a mirror image of the virtual link diagram we mean the diagram obtained from the initial one by switching all types of classical crossings (all virtual crossings stay on the same positions).

Exercise 2.3

Show that the group of the mirror virtual knot in Fig. 13 is isomorphic to $\mathbb{Z}$.

This example shows us that the knot shown above is not a classical knot. Moreover, it is a good example of the existence of a non-trivial virtual knot with group $\mathbb{Z}$ (the same as that for the unknot). The latter cannot happen in the classical case.

Strange properties of virtual knots

The simplest example of the virtual knot with group $\mathbb{Z}$ is the virtual trefoil; see Fig. 14.

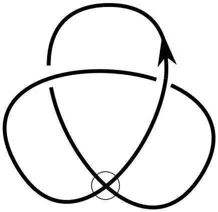

Figure 14: The virtual trefoil

Strange properties of virtual knots

The fact that the virtual trefoil is not the unknot will be proved later. Besides this example, one encounters the following strange example (Kishino knot): the connected sum of two (virtual) unknots is not trivial; see Fig. 15. We shall discuss this problem later, while speaking about long virtual knots [7].

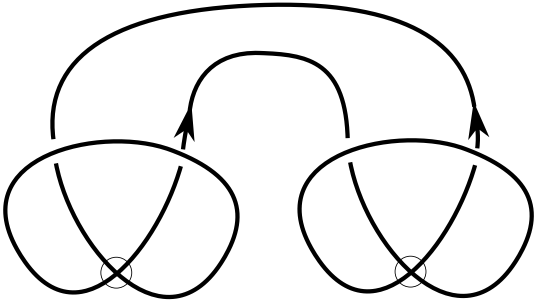

Figure 15: Non-trivial connected sum of two unknots

Strange properties of virtual knots

It is known that in the classical case long knot equivalence classes are the same as (compact) knot equivalence classes. In virtual knot theory different long knots can close up to the same knot, non-trivial long knots can close up to the unknot, and long knots almost never commute (unlike the classical case).

It is well known that the complement of each classical knot is an Eilenberg–McLane space $K(\pi, 1)$, for which all cohomology groups starting from the second group, are trivial. However, this is not the case for virtual knots: if we calculate the second cohomology the $K(\pi, 1)$ space where π is some virtual knot group, we might have some torsion. In [5] one can find a detailed description of such torsions.

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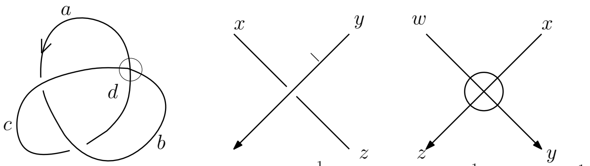
The virtual knot group G(K)

Denote all arcs of the diagram K by a_i, $i = 1, \ldots, n$. The virtual knot group G(K) is the group generated by $a_1, \ldots, a_n, q$ with relations described in Fig 16.



$z = y^{-1}xy$ $\quad z = qxq^{-1}$ $\quad w = qyq^{-1}$

$$G(K) = \langle a, b, c, d, q | b = c^{-1}ac, d = b^{-1}cb, a = qdq^{-1}, b = qcq^{-1} \rangle$$

Figure 16: The virtual knot group of the virtual trefoil knot

Let us call the quotient group of a virtual group over the relation $q = 1$ a classical group.

Theorem 2.4

The pair (the group $G(K)$, the element $q \in G(K)$) is an invariant of a virtual link. In other words, if diagrams K and K' give equivalent virtual links, then there exists an isomorphism $h : G(K) \to G(K')$ such that $h(q_K) = q_{K'}$.

It is evident that for the unknot the group is $\langle a, q \rangle$; this is the free group with two generators.

In the case of a classical diagram, we get the Wirtinger presentation with one generator q (which does not take part in the relations); thus, the group described above will be the free product of the knot group with the infinite cyclic group generated by q.

Some virtual links (and their invariants) can possess properties which do not exist in the classical case. For example, in both the classical and virtual cases we can define upper and lower groups (in the classical case we have the fundamental group of the complement); the lower group for a virtual knot is defined as the group of the mirror image of the virtual knot. For classical knots these groups are isomorphic under geometric reasons. In the virtual case the upper and lower representations can give non-isomorphic classical groups. In the examples given in the next page we give a group without an additional generator q.

Let us consider a virtual knot diagram shown in Fig. 17.

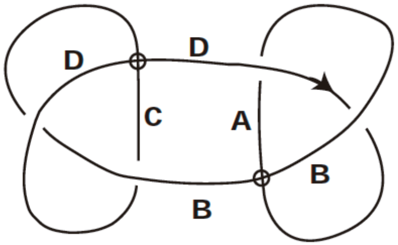


Figure 17: A virtual knot with the non-isomorphic upper and lower groups

Its four long arcs a, b, c, d shown in Fig. 17 can be chosen as generators of the group. Here a long arc goes from one undercrossing to the next undercrossing, and forms overcrossings and virtual crossings during the way.

We shall get the following relations:

$$b = dad^{-1}, a = bdb^{-1}, d = bcb^{-1}, c = dbd^{-1}.$$

Thus, the generators a and c can be expressed from the generators b and d. Therefore, we get the presentation $\langle b, d | bdb = dbd \rangle$, which represents the group isomorphic to the group of the trefoil. It is not difficult to note that the group of the mirror image of the given knot is isomorphic to the group $\mathbb{Z}$, the group of the unknot.
From the theorem of Dehn–Papakyriakopoulos [11], it follows that among non-trivial classical knots there does not exist a knot having the fundamental group of the complement isomorphic to $\mathbb{Z}$, therefore, our knot is not classical.

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Virtual knot quandle

We have defined the virtual knot group and even its generalised version (with a new generator q). Quite similarly, one defines the virtual knot quandle Q(K) which coincides with the usual quandle for the case of classical knot. From the existence of a quandle, we get the following

Theorem 3.1

If two classical links K_1, K_2 are equivalent as virtual links then they are equivalent as classical links.

Proof.

If K_1 and K_2 are virtually equivalent then $Q(K_1)$ is isomorphic to $Q(K_2)$. Since Q is a complete invariant of classical links[a] we have that K_1 is equivalent to K_2. □

[a] Actually, up to some minor equivalence, but it can be handled

The above theorem can be proved in many ways. We mention just two of them.

1. Any virtual link has a unique representative of minimal surface genus (up to isotopy) [6]. If a knot is classical then this genus is 0 and this unique representative gives rise to a links in $S^2 \times I$. It is known that classification of links in $S^2 \times I$ is the same as that in $\mathbb{R}^3$.
2. There is a well-defined projection from virtual knot Gauss diagrams to classical knot Gauss diagrams which is identical on classical diagrams. This projection agrees with Reidemeister moves. It is realised by deleting some chords of the Gauss diagram [8].

Hence, if two classical knot (Gauss) diagrams are related by a sequence of Reidemeister moves which contains virutal knot diagrams, then we can get a sequence of classical knot diagrams.

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The Jones–Kauffman polynomial

The Kauffman construction for the Jones polynomial for virtual knots [2] works just as well as in the case of classical knots. Namely, we first consider an oriented link L and the corresponding unoriented link |L|. After this, we smooth all classical crossings of |L| just as before (obtaining states of the diagram). In this way, we obtain a diagram without classical crossings, which is an unlink diagram. The number of components of this diagram (for a state s) is denoted by $\gamma(s)$. Then we define the Kauffman bracket by the same formula

$$X(L) = \sum_{s} (-a)^{3w(L)} a^{\alpha(s)-\beta(s)} (-a^2 - a^{-2})^{\gamma(s)-1},$$

where w(L) is the writhe number taken over all classical crossings of L, and $\alpha(s), \beta(s)$ are defined as in the classical case.
The invariance proof for this polynomial under classical Reidemeister moves is just the same as in the classical case; under purely virtual and semivirtual moves it is clearly invariant term–by–term.
However, this invariant has a disadvantage [2]: invariance under a move that might not be an equivalence.

Fact: Degrees of terms in the Kauffman bracket are no longer congruent mod 4. They are just congruent mod 2.

Corollary 4.1

If for a virtual link K the Kauffman bracket has terms whose exponents are not congruent mod 4, then this link is not classical.

Virtualisation

Virtualisation is the transformation which replaces a classical rossing with a classical and two virtual crossings as shown in Fig. 18 [9].

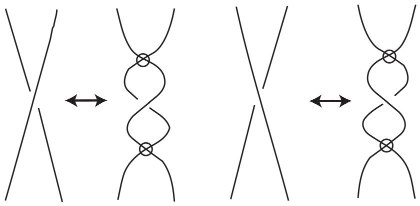

Figure 18: Two variants of virtualisation.

Theorem 4.2 ([3])

The virtualisation does not change the value of the Jones–Kauffman polynomial.

Example

There are many generalisations of the Jones polynomial for virtual knots (including the parity bracket and picture-valued invariants). The Khovanov homology for virtual knots will be constructed in the next lecture.

Let us consider the virtual knot diagram shown in Fig. 19. This knot was first considered by Kauffman.

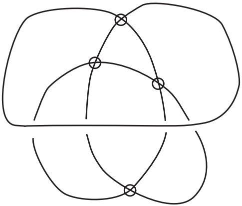

Figure 19: A virtual knot reduced to the unknot by the virtualisation and the generalised Reidemeister moves.

This knot can be reduced to the unknot with virtualisations and generalised Reidemeister moves; see Fig. 20 in the next page.

Example

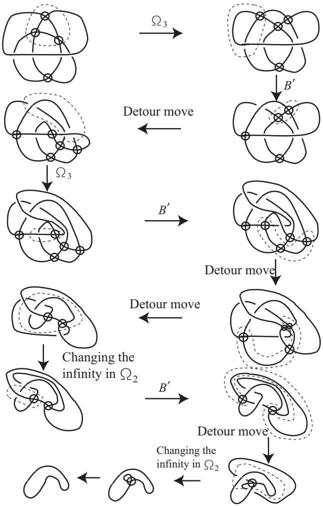

Figure 20: Reducing to the unknot by virtualisations and generalised Reidemeister moves.

Example

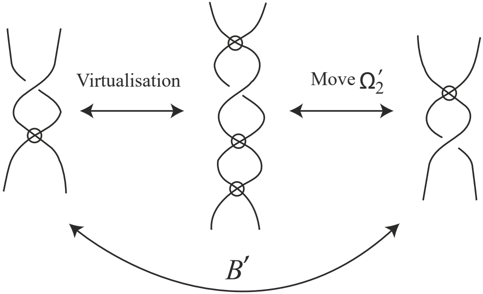

Figure 21: The move B′ is expressed in terms of the virtualisation.

In Fig. 20 by the transformation B′ we mean a move applied to one classical and one virtual crossing; it represents a composition of the virtualisation and the second Reidemeister move; see Fig. 21. For each of the transformations shown in Fig. 20, we pick out a domain which this transformation is applied to.

Thus, the Jones–Kauffman polynomial of the knot depicted in Fig. 19 coincides with the polynomial of the unknot. One can show though (e.g. using the techniques of virtual quandles) that this virtual knot is not trivial.

Other than virtual moves and Jones polynomial for virtual knots, there exist virtual braids, virtual Markov and Alexander theorem!

Contents

Exercises

1. Prove that all crossings of a realisable classical Gauss diagrams are even.
2. If a Gauss diagram of a virtual knots has only even chords then exponents of all terms in the Kauffman bracket of this knot are congruent modulo 4.
3. Let L be a virtual link two-component diagram. Assume the number of virtual mixed crossings (i.e., virtual crossings formed by two different components) is odd. Prove that the link L is not classical.
4. By looking at the Kauffman bracket of the virtual trefoil knot, show that it is not classical.

Contents

Unsolved problems

1. Construct a (formally) complete invariant of virtual knots.
2. It is known that virtual knots and links are algorithmically recognisable [9].
   Is there a constructive algorithm for virtual knot recognition?
3. There is a projection map from classical knots to virtual [?]. Are there (non-obvious) maps from classical knots to virtual knots? If there were some, one could construct absolutely new (say, picture-valued) invariants of classical knots.
4. To which extent can the methods of constructing invariants of virtual knots be generalised for knots with arbitrary manifolds?

Contents

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