Lecture No 14 April 22, 2022 (Fri)

[Proposition 12.3] (1) For $f, g \in \mathcal{L}^2_T$, we have

$$\langle I(f), I(g)\rangle_t = \int_0^t f_s\, g_s\, ds, \quad t \in [0, T].$$

In particular, Itô isometry holds:

$$E[I_t(f)^2] = E\Big[\int_0^t f_s^2\, ds\Big], \quad E[I_t(f) I_t(g)] = E\Big[\int_0^t f_s\, g_s\, ds\Big].$$

(2) For $f, g \in \mathcal{L}^2_T$ and $a, b \in \mathbb{R}$, we have

$$I_t(af + bg) = aI_t(f) + bI_t(g), \ {}^\forall t \in [0, T] \ \text{ a.s.}$$

[Remark 12.2] Note that the exceptional set (i.e. the set of ω for which (2) does not hold) depends on the choice of (f, g, a, b). Since it's uncountably many, we cannot take the exceptional set common in (f, g, a, b). Thus, (2) is different from the linearity of the usual integrals. □

[Proof] • (1) was shown for $f, g \in \mathcal{S}_T$. In particular, for $\Phi = \Phi(\omega)$: $\mathcal{F}_s$-measurable bounded function, we have

$$E\left[\left\{I_t(f)I_t(g) - I_s(f)I_s(g) - \int_s^t f_r g_r \, dr\right\}\Phi\right] = 0$$

Take sequences $f^n \in \mathcal{S}_T \to f$, $g^n \in \mathcal{S}_T \to g$ in $\mathbb{L}^2_T$. Then, the above identity holds for f^n, g^n. Taking the limit $n \to \infty$, we obtain it also for $f, g \in \mathcal{L}^2_T$.

• (2) is obvious for $f, g \in \mathcal{S}_T$ by definition. For $f, g \in \mathcal{L}^2_T$, we may take approximating sequences and take the limit $n \to \infty$ again. □

Extension to the time interval $[0, \infty)$.

- Since $^\forall T > 0$ was taken arbitrarily, setting

$$\mathcal{L}^2 := \{f = (f_t)_{t\geq 0};\ (f_t)_{t\in[0,T]} \in \mathcal{L}^2_T \text{ for } {}^\forall T > 0\},$$

  one can define stochastic integral $\int_0^t f_s\, dB_s,\ t \geq 0$ for $f \in \mathcal{L}^2$.
- Note that it is well-defined independently of the choice of T, that is, for $0 < T_1 < T_2$, stochastic integral defined on $[0, T_2]$ and restricted to $[0, T_1]$ coincides a.s. with that defined on $[0, T_1]$. Indeed, this is true when f is a step process and we may take the limit. □

Stochastic integrals w.r.t. d-dimensional Brownian motion.

- Let $B = (B^i)_{1 \le i \le d}$ be a d-dimensional $(\mathcal{F}_t)$-Brownian motion i.e. each component B^i is $(\mathcal{F}_t)$-Brownian motion and $\{B^i\}_{1 \le i \le d}$ is $\perp\!\!\!\perp$.
- Then, for $f \in \mathcal{L}^2$, we can define the stochastic integral

$$I_t^i(f) = \int_0^t f_s \, dB_s^i, \;\; t \ge 0, \; 1 \le i \le d.$$

[Proposition 12.4] For $f, g \in \mathcal{L}_T^2$, we have

$$\langle I^i(f), I^j(g) \rangle_t = \delta_{ij} \int_0^t f_s g_s \, ds, \quad t \in [0, T],$$

where δ_{ij} is Kronecker's δ. In particular,

$$E\left[\int_0^t f_s \, dB_s^i \int_0^t g_s \, dB_s^j \right] = \delta_{ij} E\left[\int_0^t f_s g_s \, ds \right] \qquad \square$$

[Proof] • The case $i = j$ was shown in Proposition 12.3-(1).

• Thus, we may consider the case $i \neq j$ and assume $f, g \in \mathcal{S}_T$ by approximation. As before, we may assume $f_t = f 1_{[a,b)}(t), g_t = g 1_{[c,d)}(t)$ with f:$\mathcal{F}_a$-measurable and bounded, g:$\mathcal{F}_c$-measurable and bounded, and may consider the case $[a, b) = [c, d)$ or $[a, b) \cap [c, d) = \emptyset$, since one may consider finer division for $f, g \in \mathcal{S}_T$.

• For example, in the case of $[a, b) = [c, d)$, we have

$$
\begin{aligned}
&E[I_t^i(f) I_t^j(g) - I_s^i(f) I_s^j(g) | \mathcal{F}_s] \\
&= E[fg\{(B^i_{t\wedge b} - B^i_{t\wedge a})(B^j_{t\wedge b} - B^j_{t\wedge a}) - (B^i_{s\wedge b} - B^i_{s\wedge a})(B^j_{s\wedge b} - B^j_{s\wedge a})\} | \mathcal{F}_s]
\end{aligned}
$$

and, as before, we may compute separately for $s \leq a$ and $s > a$, and obtain $= 0$ by the independence of Brownian motion for $i \neq j$. □

P: Show the above more carefully.

In particular, taking $f = g = 1$ in Proposition 12.4, we have

[Corollary 12.5] $\langle B^i, B^j \rangle_t = \delta_{ij} t$ □

In fact, it is known that the converse is also true
(see Karatzas-Shreve [3]):

[Theorem 12.6] (Lévy's theorem: characterization of Brownian motion by martingales) Let $M_t = (M_t^i)_{i=1}^d$ be given and assume $M^i \in \mathcal{M}_{c,T}^2 = \mathcal{M}_{c,T}^2(\mathcal{F}_t)$ and $\langle M^i, M^j \rangle_t = \delta_{ij} t$ holds. Then, M_t is a d-dimensional $(\mathcal{F}_t)$-Brownian motion. □

[Remark 12.3] (1) (see Karatzas-Shreve [3], p.146)
The condition $f \in \mathcal{L}^2_T$ (in particular, L^2-integrability on $[0, T] \times \Omega$) for the integrand f can be relaxed as

$$\int_0^T f_t^2(\omega)\,dt < \infty \quad \text{a.s.}$$

We need to localize in time by considering Markov times and introduce the notion of local martigales.

(2) In general, we can define stochastic integrals $\int_0^t f_s dM_s$ with respect to $(\mathcal{F}_t)$-martingales $M_t \in \mathcal{M}^2_{c,T}$ instead of Brownian motion. It is a continuous version of the martingale transform $\sum_{k=2}^n H_k(M_k - M_{k-1})$ for a predictable H in the discrete time setting. It is more desirable to consider such stochastic integrals to provide a good mathematical framework. □

§13 Itô's formula

- For $\varphi = \varphi(x), x = x_t \,(= x(t)) \in C^1(\mathbb{R})$, the following differential formula for composite functions holds:

  $$\frac{d}{dt}\varphi(x_t) = \varphi'(x_t)\frac{dx_t}{dt}$$

  or, in integrated form, we have

  $$\varphi(x_t) = \varphi(x_0) + \int_0^t \varphi'(x_s)dx_s,$$

  where RHS is defined as a Stieltjes integral, which is definable if x_t is of bounded variation in t.
- [Question] Is this formula still true, if we take a Brownian motion B_t for x_t and understand RHS as a stochastic integral?
  [Answer] A second order correction term is required and

  $$\varphi(B_t) = \varphi(B_0) + \int_0^t \varphi'(B_s)dB_s + \frac{1}{2}\int_0^t \varphi''(B_s)ds$$

  holds. This is called Itô's formula.

[Example 13.1] • For $\varphi(x) = x^2$, $B_t^2 = B_0^2 + \int_0^t 2B_s dB_s$ does not hold. Indeed, since the stochastic integral is a martingale and $B_0 = 0$, we see that $E[\text{RHS}] = 0$. However, for LHS, $E[B_t^2] = t$ so that the above formula is not true.

• Let $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ be a division of the interval $[0, t]$. Recalling that we take the value of the integrand at the left edge of each small interval in the definition of stochastic integral, and also recalling the computation we made for the quadratic variation, we obtain

$$\begin{aligned}\int_0^t B_s dB_s &= \lim_{|\Delta|\to 0} \sum_{j=1}^{n} B_{t_{j-1}}(B_{t_j} - B_{t_{j-1}}) \quad (\text{in } L^2(\Omega)) \\ &= \tfrac{1}{2} \lim_{|\Delta|\to 0} \sum_{j=1}^{n} \{(B_{t_j}^2 - B_{t_{j-1}}^2) - (B_{t_j} - B_{t_{j-1}})^2\} \\ &= \tfrac{1}{2}\{(B_t^2 - B_0^2) - t\}\end{aligned}$$

Therefore, we have

$$B_t^2 - B_0^2 = 2\int_0^t B_s dB_s + t.$$

This shows a correction of the second order is required. □

- Setting to state Itô's formula:

$X_t = (X_t^i)_{i=1}^d$: d-dimensional continuous stochastic process and each component has the form:

$$X_t^i = X_0^i + \sum_{k=1}^{n} \int_0^t f_k^i(s)\, dB_s^k + A_t^i, \quad t \in [0, T], \qquad (1)$$

where

- $B_t = (B_t^k)_{k=1}^n$: n-dimensional $(\mathcal{F}_t)$-Brownian motion
- X_0^i : $\mathcal{F}_0$-measurable r.v., $1 \le i \le d$
- $f_k^i = (f_k^i(t)) \in \mathcal{L}_T^2 = \mathcal{L}_T^2(\mathcal{F}_t)$, $1 \le k \le n, 1 \le i \le d$
- A_t^i : $(\mathcal{F}_t)$-adapted continuous stochastic process s.t. $A_0^i = 0$ and, for a.s. ω, $A_t^i(\omega)$ are of bounded variation in t.

- Stochastic process of the form (1) is called Itô process.
- In general, stochastic processes of the form $X = M + A$ (M: martingale) is called semi-martingale.

Let $X_t = (X_t^i)_{i=1}^d$, $t \in [0, T]$ be given as above.

$C_b^2(\mathbb{R}^d) := \{\varphi \in C^2(\mathbb{R}^d);\ \varphi$ itself and its derivatives up to second order are all bounded$\}$.

[Theorem 13.1] (Itô's formula) For $^\forall \varphi \in C_b^2(\mathbb{R}^d)$, we have

$$\begin{aligned}\varphi(X_t) =& \varphi(X_0) + \sum_{i=1}^{d}\sum_{k=1}^{n}\int_0^t \frac{\partial \varphi}{\partial x^i}(X_s) f_k^i(s)\, dB_s^k \\ &+ \sum_{i=1}^{d}\int_0^t \frac{\partial \varphi}{\partial x^i}(X_s)\, dA_s^i \\ &+ \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{n}\int_0^t \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(X_s) f_k^i(s) f_k^j(s)\, ds,\end{aligned}$$

for $t \in [0, T]$ a.s. □

[Remark 13.1] (Interpretation, Actual way of computation)
We write Itô process of the form (1) formally in terms of the stochastic differential:

$$dX_t^i = f_k^i(t)\, dB_t^k + dA_t^i.$$

We omit $\sum_k$ using Einstein convention. Then, by Taylor expansion,

$$d\varphi(X_t) = \frac{\partial \varphi}{\partial x^i}(X_t)\, dX_t^i + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^i \partial x^j}(X_t)\, dX_t^i dX_t^j + \cdots$$

We apply the following rule to compute the product of stochastic differentials $dX_t^i dX_t^j$:

$$\begin{cases} dB_t^k dB_t^{k'} = \delta^{kk'} dt & \longleftrightarrow \quad \langle B^k, B^{k'} \rangle_t = \delta^{kk'} t \\ dB_t^k dA_t^i = dA_t^i dA_t^j = 0. \end{cases}$$

Roughly saying, we regard $dB_t^k \sim \sqrt{dt}$, $dA_t^i \sim dt$ and ignore terms of order smaller that dt.

By this rule, we have

$$\begin{aligned} dX_t^i dX_t^j &= (f_k^i(t)\, dB_t^k + dA_t^i)(f_{k'}^j(t)\, dB_t^{k'} + dA_t^j) \\ &= \sum_{k=1}^n f_k^i(t) f_k^j(t)\, dt \\ dX_t^i dX_t^j dX_t^\ell &= 0. \end{aligned}$$

Therefore, up to second order terms in Taylor expansion survive and we have

$$\begin{aligned} d\varphi(X_t) =& \frac{\partial \varphi}{\partial x^i}(X_t) f_k^i(t)\, dB_t^k + \frac{\partial \varphi}{\partial x^i}(X_t)\, dA_t^i \\ &+ \frac{1}{2} \sum_{k=1}^n \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(X_t) f_k^i(t) f_k^j(t)\, dt. \end{aligned}$$

Writing this in an integrated form, we obtain Theorem 13.1 (Itô's formula). □

Since stochastic differential has only a formal meaning, mathematically, we need to write Theorem by means of stochastic integrals.

[Outline of proof of Theorem 13.1]

[Step 1] We only consider the case $n = d = 1$ for simplicity. Since the formula is usual in terms of the bounded-variation part A_t, we assume $A_t = 0$. Namely, for $\mathcal{F}_0$-measurable r.v. X_0 and $f = (f(t)) \in \mathcal{L}^2_T$, we define $\mathbb{R}$-valued process X_t as

$$X_t = X_0 + \int_0^t f(s)\,dB_s, \quad t \in [0, T], \tag{1}$$

and show for $\varphi \in C^2_b(\mathbb{R})$ and $t \in [0, T]$

$$\varphi(X_t) = \varphi(X_0) + \int_0^t \varphi'(X_s) f(s)\,dB_s + \frac{1}{2}\int_0^t \varphi''(X_s) f^2(s)\,ds. \tag{2}$$

([Remark] Under this setting, we only see $dBdB = dt$, but not $dB^k dA^i = 0$ nor $dB^k dB^{k'} = 0$ for $k \neq k'$. We restrict the case for simplicity.)

• Similarly as before, once we can show the formula (2) for $f \in \mathcal{S}_T$ (i.e. step processes), then it can be shown for $f \in \mathcal{L}_T^2$ by approximation. Therefore, we assume that f is given by

$$f(s,\omega) = \sum_{j=1}^{N} f_j(\omega) 1_{[t_{j-1},t_j)}(s), \quad s \in [0,T], \tag{3}$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ and f_j are $\mathcal{F}_{t_{j-1}}$-measurable and bounded r.v.'s.

• To show (2) for f given by (3), it is enough to show

$$\varphi(X_t) - \varphi(X_s) = \int_s^t \varphi'(X_r) f(r)\, dB_r + \frac{1}{2} \int_s^t \varphi''(X_r) f^2(r)\, dr \tag{4}$$

for s, t satisfying $t_{m-1} \le s < t \le t_m$ $(1 \le^{\exists} m \le N)$ (i.e. for s, t belonging to the same small time interval).

☺ For t such that $t_{m-1} \le t \le t_m$, we can write

$$\varphi(X_t) - \varphi(X_0) = \left(\varphi(X_t) - \varphi(X_{t_{m-1}})\right) + \sum_{j=1}^{m-1} \left(\varphi(X_{t_j}) - \varphi(X_{t_{j-1}})\right)$$

so that we may apply (4) repeatedly to each small time interval. □

- In (4), $r \in [s,t) \subset [t_{m-1}, t_m)$ and

$$f(r) \underset{(3)}{=} f(t_{m-1}) =: f$$

is $\mathcal{F}_s$-measurable and bounded, since f is actually $\mathcal{F}_{t_{m-1}}$-measurable and $t_{m-1} \leq s$ holds.

- Moreover, by (1),

$$X_t = X_s + \int_s^t f(r)\,dB_r = X_s + \int_s^t f\,dB_r = X_s + f(B_t - B_s)$$

- Summarizing these, it is enough to show

$$\varphi(X_t) - \varphi(X_s) = \int_s^t \varphi'(X_r) f\,dB_r + \frac{1}{2}\int_s^t \varphi''(X_r) f^2\,dr \qquad (4')$$

for an $\mathcal{F}_s$-measurable bounded r.v. f and X given by

$$X_t = X_s + f(B_t - B_s), \quad 0 \leq s < t \leq T. \qquad (1')$$

[Step 2] Divide the interval $[s,t]$ into n equal small pieces and set

$$t_\ell = s + \frac{t-s}{n}\ell, \qquad 0 \le \ell \le n.$$

Then, by Taylor's formula, there exists Y_ℓ between $X_{t_{\ell-1}}$ and X_{t_ℓ} such that

$$\begin{aligned}
\varphi(X_t) - \varphi(X_s) &= \sum_{\ell=1}^{n} \{\varphi(X_{t_\ell}) - \varphi(X_{t_{\ell-1}})\} \\
&= \sum_{\ell=1}^{n} \varphi'(X_{t_{\ell-1}})(X_{t_\ell} - X_{t_{\ell-1}}) + \frac{1}{2}\sum_{\ell=1}^{n} \varphi''(Y_\ell)(X_{t_\ell} - X_{t_{\ell-1}})^2 \\
&\underset{(1')}{=} \sum_{\ell=1}^{n} \varphi'(X_{t_{\ell-1}}) f (B_{t_\ell} - B_{t_{\ell-1}}) + \frac{1}{2}\sum_{\ell=1}^{n} \varphi''(Y_\ell) f^2 (B_{t_\ell} - B_{t_{\ell-1}})^2 \\
&=: I_n^{(1)} + I_n^{(2)}
\end{aligned}$$

holds. We will show that $I_n^{(1)}$ and $I_n^{(2)}$ converge to the first and second terms of (4'), respectively, as $n \to \infty$.

[Step 3] Here, we show $I_n^{(1)} = \sum_{\ell=1}^{n} \varphi'(X_{t_{\ell-1}}) f(B_{t_\ell} - B_{t_{\ell-1}}) \xrightarrow[n\to\infty]{} \int_s^t \varphi'(X_r) f \, dB_r$ in $L^2(\Omega)$.

∵ $I_n^{(1)}$ can be regarded as a stochastic integral of a step process: $I_n^{(1)} = \int_s^t \sum_{\ell=1}^{n} \varphi'(X_{t_{\ell-1}}) f 1_{[t_{\ell-1}, t_\ell)}(r) \, dB_r$. Therefore by Itô isometry, we have

$$E\left[\left\{ I_n^{(1)} - \int_s^t \varphi'(X_r) f \, dB_r \right\}^2\right]$$
$$= E\Big[\int_s^t \Big\{ \sum_{\ell=1}^{n} \varphi'(X_{t_{\ell-1}}) 1_{[t_{\ell-1}, t_\ell)}(r) - \varphi'(X_r) 1_{[s,t)}(r) \Big\}^2 f^2 dr \Big].$$

However, noting that f is bounded, φ' is bounded continuous and X_r is continuous in r, Lebesgue's convergence theorem proves that (RHS) $\to 0$ as $n \to \infty$. □

[Step 4] Finally, we show $I_n^{(2)} = \frac{1}{2}\sum_{\ell=1}^{n} \varphi''(Y_\ell) f^2 (B_{t_\ell} - B_{t_{\ell-1}})^2$
$\underset{n\to\infty}{\longrightarrow} \frac{1}{2}\int_s^t \varphi''(X_r) f^2\, dr$ in $L^2(\Omega)$.

☺ • Since f^2 is bounded, it is enough to show that

$$\sum_{\ell=1}^{n} \varphi''(Y_\ell)(B_{t_\ell} - B_{t_{\ell-1}})^2 \to \int_s^t \varphi''(X_r)\, dr \quad \text{in} \;\; L^2(\Omega),$$

by excluding $\frac{1}{2} f^2$. However, we have

$$\sum_{\ell=1}^{n} \varphi''(Y_\ell)(t_\ell - t_{\ell-1}) \to \int_s^t \varphi''(X_r)\, dr \quad \text{in} \;\; L^2(\Omega).$$

(☺ Indeed, LHS is the Riemann sum for RHS. Noting that φ'' is bounded continuous and X_r is continuous, we may apply Lebesgue's convergence theorem.)

• From these observation, it is enough to show

$$\sum_{\ell=1}^{n} \varphi''(Y_\ell)\big\{(B_{t_\ell} - B_{t_{\ell-1}})^2 - (t_\ell - t_{\ell-1})\big\} \to 0 \quad \text{in} \;\; L^2(\Omega).$$

• In other words, setting $Z_\ell := (B_{t_\ell} - B_{t_{\ell-1}})^2 - (t_\ell - t_{\ell-1})$, it is enough to prove

$$\sum_{\ell=1}^{n} \varphi''(Y_\ell) Z_\ell \to 0 \quad \text{in} \;\; L^2(\Omega).$$

• We expand and set

$$\begin{aligned} & E\Big[\Big\{ \sum_{\ell=1}^{n} \varphi''(Y_\ell) Z_\ell \Big\}^2\Big] \\ & = \sum_{\ell=1}^{n} E[\varphi''(Y_\ell)^2 Z_\ell^2] + 2 \sum_{1 \le \ell_1 < \ell_2 \le n} E[\varphi''(Y_{\ell_1}) Z_{\ell_1} \varphi''(Y_{\ell_2}) Z_{\ell_2}] \\ & =: ① + ②. \end{aligned}$$

• For ①: noting $E[Z_\ell^2] = 2(t_\ell - t_{\ell-1})^2$ as we showed before, we can estimate and obtain

$$0 \leq ① \leq \|\varphi''\|_\infty^2 \sum_{\ell=1}^{n} E[Z_\ell^2] \underset{n\to\infty}{\longrightarrow} 0$$

• For ②: First, note that, if we replace Y_{ℓ_2} by $X_{t_{\ell_2 - 1}}$, we have

$$E[\varphi''(Y_{\ell_1})Z_{\ell_1}\varphi''(X_{t_{\ell_2-1}})Z_{\ell_2}] = 0.$$

Indeed, $\varphi''(Y_{\ell_1})Z_{\ell_1}\varphi''(X_{t_{\ell_2-1}})$ is $\mathcal{F}_{t_{\ell_2-1}}$-measurable, so that it is $\perp\!\!\!\perp Z_{\ell_2}$. Therefore, the above is shown from $E[Z_{\ell_2}] = 0$.

- Accordingly, to prove ② $\to 0$, it is enough to show

$$\sum_{1\leq \ell_1<\ell_2\leq n} E[\varphi''(Y_{\ell_1})Z_{\ell_1}\{\varphi''(Y_{\ell_2})-\varphi''(X_{t_{\ell_2-1}})\}Z_{\ell_2}] \to 0$$

However, estimating the first φ'' by $\|\varphi''\|_\infty$, moving it outside of the expectation, and then applying Schwarz's inequality, we see that the above sum is bounded by

$$\|\varphi''\|_\infty \sum_{1\leq \ell_1<\ell_2\leq n} E\big[Z_{\ell_1}^2 Z_{\ell_2}^2\big]^{\frac{1}{2}} \cdot E\big[\{\varphi''(Y_{\ell_2})-\varphi''(X_{t_{\ell_2-1}})\}^2\big]^{\frac{1}{2}}$$

However, since $Z_{\ell_1} \perp\!\!\!\perp Z_{\ell_2}$, we see

$$E\big[Z_{\ell_1}^2 Z_{\ell_2}^2\big] = E\big[Z_{\ell_1}^2\big]E\big[Z_{\ell_2}^2\big] = 4(t_{\ell_1}-t_{\ell_1-1})^2(t_{\ell_2}-t_{\ell_2-1})^2$$

On the other hand, the second expectation is bounded by

$$E\big[\max_{1\leq \ell\leq n}\{\varphi''(Y_\ell)-\varphi''(X_{t_{\ell-1}})\}^2\big]^{\frac{1}{2}},$$

which does not depend on ℓ_1, ℓ_2, so that one can move it outside of the sum. For the remaining sum, we have

$$\sum_{1\leq \ell_1<\ell_2\leq n} 2(t_{\ell_1}-t_{\ell_1-1})(t_{\ell_2}-t_{\ell_2-1}) \leq T^2.$$

- Summarizing these, to prove ② $\to 0$, it is enough to show

$$E\Big[\max_{1\le\ell\le n}\{\varphi''(Y_\ell)-\varphi''(X_{t_{\ell-1}})\}^2\Big]\to 0$$

However , since φ'' is bounded continuous and X_r is continuous in r, Lebesgue's convergence theorem shows this.

Thus, the proof of Itô's formula, in the simplest setting, is completed. □