A nontrivial example that can be solved exactly

The Ising chain

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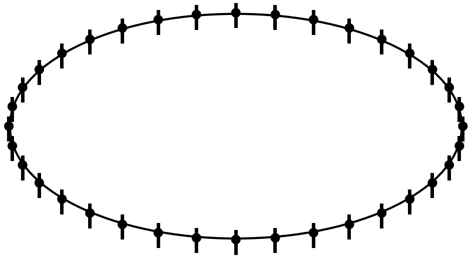

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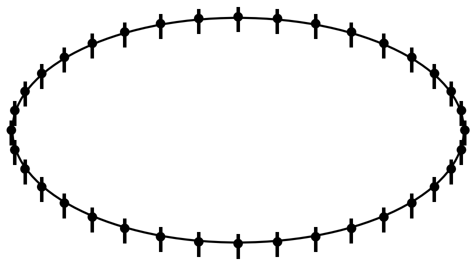
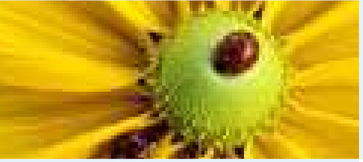


- The spins then satisfy the periodic boundary conditions

$$\sigma_i = \sigma_{i+N} ,$$

and the energies of the nearest-neighbor Ising Hamiltonian are

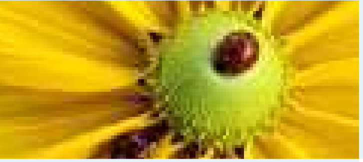
$$E_\sigma = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i .$$



Partition function

- The second term can be rewritten so that it has the same symmetry as the first term:

$$E_\sigma = -J \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - \frac{h}{2} \sum_{i=1}^{N} (\sigma_i + \sigma_{i+1}).$$



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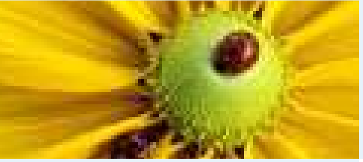
- The partition function of the Ising chain is then

$$\begin{aligned} Z_C &= \sum_\sigma \exp(-\beta E_\sigma) \\ &= \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_N=\pm 1} \exp\left\{ \beta \sum_{i=1}^{N} \left[J\sigma_i \sigma_{i+1} + \frac{h}{2}(\sigma_i + \sigma_{i+1}) \right] \right\} \\ &= \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_N=\pm 1} \prod_{i=1}^{N} \exp\left\{ \beta \left[J\sigma_i \sigma_{i+1} + \frac{h}{2}(\sigma_i + \sigma_{i+1}) \right] \right\}. \end{aligned}$$

Transfer matrix

- Let us now define a 2×2 matrix P with matrix elements

$$P_{\sigma_i \sigma_j} = \langle \sigma_i | P | \sigma_j \rangle = \exp\left\{ \beta \left[J \sigma_i \sigma_j + \frac{h}{2} (\sigma_i + \sigma_j) \right] \right\}.$$



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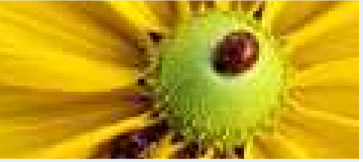
- In particular, we have

$$\langle + | P | + \rangle = \exp[\beta(J + h)], \qquad \langle - | P | - \rangle = \exp[\beta(J - h)],$$
$$\langle + | P | - \rangle = \langle - | P | + \rangle = \exp(-\beta J),$$

hence

$$P = \begin{pmatrix} \mathrm{e}^{\beta(J+h)} & \mathrm{e}^{-\beta J} \\ \mathrm{e}^{-\beta J} & \mathrm{e}^{\beta(J-h)} \end{pmatrix}.$$



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- In terms of this matrix, the partition function is

$$Z_C = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \prod_{i=1}^{N} \langle \sigma_i | P | \sigma_{i+1} \rangle$$
$$= \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} \langle \sigma_1 | P | \sigma_2 \rangle \langle \sigma_2 | P | \sigma_3 \rangle \cdots \langle \sigma_N | P | \sigma_{N+1} \rangle.$$

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- The trace is easiest to evaluate in a basis where P is diagonal.
- Let λ_1 and λ_2 be the eigenvalues of P; i.e., the solutions to

$$\det(P - \lambda I) = 0.$$

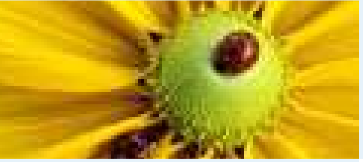
- In a basis of its own eigenvectors, P has the form

$$P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \Rightarrow \qquad P^N = \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix}.$$

Free energy

- The trace of P^N is then just

$$\mathrm{tr}(P^N) = \lambda_1^N + \lambda_2^N .$$



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- Choosing λ_1 to be the larger eigenvalue, we have

$$Z_C = \lambda_1^N [1 + (\lambda_2/\lambda_1)^N],$$

and the magnetic free energy is

$$\begin{aligned}\tilde{A} &= -k_\mathrm{B} T \log Z_C \\ &= -N k_\mathrm{B} T \log \lambda_1 - k_\mathrm{B} T \log[1 + (\lambda_2/\lambda_1)^N].\end{aligned}$$

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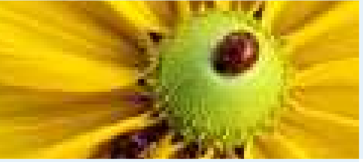
- Since $\lambda_2 < \lambda_1$, the second term vanishes in the thermodynamic limit $N \to \infty$, and we have simply[1]

$$\tilde{A} = -N k_{\mathrm{B}} T \log \lambda_1.$$

- Hence, our last remaining problem is to find the eigenvalues of P.

[1] Exercise: Show that you get the same result for $\tilde{A}$ even when $\lambda_2 = \lambda_1$.



Eigenvalues

- To find the eigenvalues of P, we need to solve the secular equation

$$\det \begin{pmatrix} \mathrm{e}^{\beta(J+h)} - \lambda & \mathrm{e}^{-\beta J} \\ \mathrm{e}^{-\beta J} & \mathrm{e}^{\beta(J-h)} - \lambda \end{pmatrix} = 0.$$

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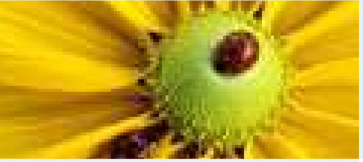

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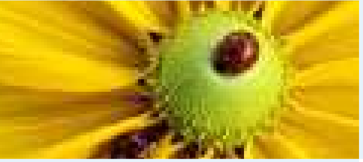
- In the special case $h = 0$, the solutions reduce to

$$\lambda_1 = 2\cosh \beta J, \qquad \lambda_2 = 2\sinh \beta J.$$

Free energy solution

- Hence, the free energy in the thermodynamic limit is

$$\tilde{A}(T,h) = -N k_{\mathrm{B}} T \log\left[\mathrm{e}^{\beta J}\left(\cosh\beta h + \sqrt{\sinh^2\beta h + \mathrm{e}^{-4\beta J}}\right)\right].$$



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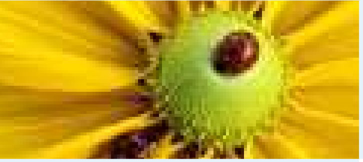

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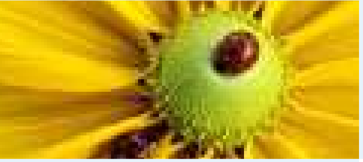
- This closed-form solution is completely regular (except at $T = 0$) and analytic in both T and h.
- There is consequently no non-analytic behavior (at finite T) in any of the physically meaningful thermodynamic variables such as

$$M = -\left(\frac{\partial \tilde{A}}{\partial B} \right)_T, \qquad \chi_T(B) = \frac{\mu_0}{V} \left(\frac{\partial M}{\partial B} \right)_T.$$

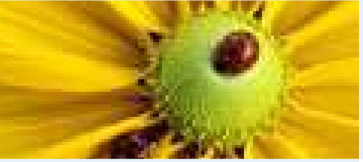


Mean-field theory

- However, our mean-field calculations predicted a second-order phase transition in $M(T)$ at the nonzero temperature

$$T_c = qJ/k_{\mathrm{B}},$$

where $q = 2$ in one dimension.



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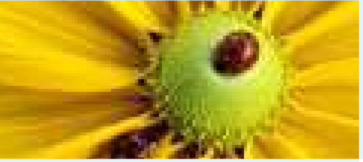

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- Conclusion: Fluctuation effects can be very important, especially in reduced dimensions.
- Even in 3D, fluctuations change the magnetization near T_c in a nontrivial way, since although mean-field theory does correctly predict

$$m(T) \propto (T_c/T - 1)^\beta,$$

it predicts a critical exponent $\beta = 1/2$, which disagrees with the measured value $\beta = 0.334$.