2020-10-18 Kähler geometry

**Lemma** $v \in \mathrm{Sym}(T_J'M \otimes T_J'M)$ ①

symmetric part.

**Proof** A symplectic form $\omega$ on $T_J'M$ defines a symplectic form $\omega^{-1}$ on $T_J^{*\prime}M$.

If $\alpha, \beta \in T_J^{*\prime}M$, then by $J$-invariance of $\omega^{-1}$

$$\omega^{-1}(\alpha, \beta) = \omega^{-1}(J\alpha, J\beta) = \omega^{-1}(i\alpha, i\beta)$$

$$= -\omega^{-1}(\alpha, \beta)$$

$$\therefore \omega^{-1}(\alpha, \beta) = 0.$$

Similarly
$$\omega^{-1}(v\alpha, v\beta) = 0.$$

Since $\omega^{-1}$ is also $J'$-invariant

$$\omega^{-1}(\alpha + v(\alpha), \beta + v(\beta)) = 0.$$

Thus
$$\omega^{-1}(\alpha, v(\beta)) = -\omega^{-1}(v(\alpha), \beta)$$

$$= \omega^{-1}(\beta, v(\alpha)) \quad v^{ji}$$

$$v^{ij} \quad \underbrace{g^{j\bar{k}}}\, \alpha_j \, \underbrace{v^i{}_{\bar{k}}}\, \beta_i = \underbrace{g^{i\bar{k}}}\, \beta_i \, \underbrace{v^j{}_{\bar{k}}}\, \alpha_j.$$

$v^{ij} = v^{ji}$

$S_0$, $T_J Z \subset C^\infty(\mathrm{Sym}(T_J^{\prime} M \otimes T_J^{\prime} M))$

$L^2$-inner product.
$i$ complex str.
$\Rightarrow$ $Z$ : Kähler str.

$K$ = the group of all Hamiltonian symplectomorphisms of $(M, \omega)$

$\mathfrak{k} = C_0^\infty(M) = \{u \in C^\infty(M) \mid \int_M u\,\omega^n = 0\}$

$i(X)\omega = -du_X$

"normalized Hamiltonian functions"

$K$ acts on $Z$ as holomorphic isometries.

$f \in K$

$f : (M, \omega, J) \longrightarrow (M, \omega, J)$

$f^*J$

$\mu$ $\int_M \mu u \omega^n$

$\int_M (\mu + c) u\,\omega^n$

<u>Theorem</u> (Donaldson–Fujiki)

$$\mu : Z \longrightarrow C^\infty(M)/\mathbb{R} = \mathfrak{k}^*$$

$$J \longmapsto (S(J), \cdot)_{L^2}$$

$L^2$-dual of scalar curvature $S(\omega, J) =: S(J)$

is an equivariant moment map.

<u>Proof omitted.</u>

$$\mu^{-1}(0) = \{ J \mid (M, \omega, J) \text{ has a constant scalar curvature} \}$$

If we apply Kempf-Ness naively in this infinite dimensional setting, the problem can be regarded as a GIT problem.

<u>Remark</u> $K^c$ does not exist in this infinite dimensional setting.

But $k^c = \{ u + iv \mid u, v \in C_0^\infty(M) \}$ exists. This defines a foliation of $\mathcal{J}$. Each leaf plays ~~the~~ role of $K^c$-orbit $\Gamma$ and $\Gamma/K$ is diffeomorphic to the space of Kähler forms in a fixed Kähler class.

( homework : hint

$L_{JX}\omega = i\partial\bar\partial v_X$

$X$ Hamiltonian $\to v_X$ )

$\omega \to \omega + i\partial\bar\partial\varphi$

Work of Xiaowei Wang:

Moment maps, Futaki invariant and stability of projective manifolds, Comm. Anal. Geom. 12 (2004), 187-196.

° Lichnerowicz-Matsushima, Futaki can be both interpreted as obstructions to stability in terms of finite dimensional moment map picture. ($\exists$ finite dimensional analogues.)

$$\begin{array}{ccl} \Lambda \longrightarrow & N & \text{f.n.dim ample line bundle.} \\ & \downarrow & \\ \mathcal{L} \longrightarrow & Z & \text{infinite di.} \end{array}$$

$$\overset{\text{di}}{\wedge} \operatorname{Ker} P_p \otimes \left(\overset{\text{di}}{\wedge} \operatorname{Im} P_p\right)^*$$

Tian (1997)

$\exists \mathcal{L} \rightarrow$ space of Kähler forms as the determinant line bdle of some family of elliptic operators such that log(Quillen metric) = Mabuchi K-energy ⑤

"analytic Torsion"
W. Müller - K. Wendland

"CM-line bundle"

Question : Properness of K-energy.

Tian : The behavior of K-energy at infinity is approximated by "Futaki invariant".

Need to the degeneration of M.
normal varieties Ding-Tian defined generalized Futaki invariant
"special degeneration" (1997)

Alg geometers thought "normality condition" is a drawback because degeneration of alg variety is not normal in general.
2002. Donaldson defined "test configuration"
"Donaldson-Futaki inv" without "normality".

Li (Chi) - Xu (Chenyang) 

Starting without normality, we can reduce the test configuration with normal central fiber (degeneration). Using Minimal model program.

As a result, Tian's definition was OK for Fano manifold.

We go back to finite dim moment map geometry.

Question: How can we check properness of $H = \log |r|^2$ $\quad r \in \underset{\mathbb{C}^*\text{-orbit}}{P} \subset \Lambda^{-1}$.

The answer (Mumford) is Hilbert-Mumford criterion.

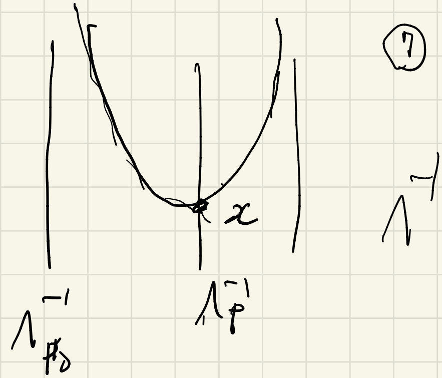
$$\Lambda^{-1} \longrightarrow \underset{\circlearrowleft}{N} \quad \text{with } \mathbb{C}^*\text{-action.}$$

$$\begin{aligned} \sigma : \mathbb{C}^* &\longrightarrow K^{\mathbb{C}} \\ t &\longmapsto \sigma(t) \end{aligned}$$

If $\lim_{t \to 0} (\sigma(t) p) = p_0$

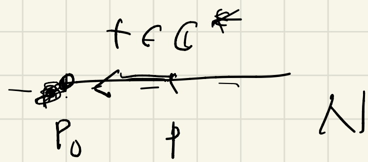
then

$$\sigma(t) p_0 = p \quad \text{for } \forall t$$




$$\sigma(t) \Lambda^{-1}_{p_0} = \Lambda^{-1}_{p_0}$$

1-dim




$$\begin{array}{rccc} \sigma(t) : & \Lambda^{-1}_{p_0} & \longrightarrow & \Lambda^{-1}_{p_0} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & z & \longmapsto & t^{-\alpha} z \end{array}$$

$$\Lambda^{-1}_{p} \ni x$$

$$\{ \sigma(t) x \mid t \in \mathbb{C}^* \} \text{ is closed}$$

$$\Longleftrightarrow \quad H = \log |r|^2 \text{ is proper.}$$

$$\Longleftrightarrow \quad t^{-\alpha} z \longrightarrow \infty \quad \text{as } t \to 0$$

$$\Longleftrightarrow \quad \alpha > 0.$$

<u>Def</u> $\alpha$ is called <u>Mumford weight</u>.

Hilbert-Mumford criterion

$\mathbb{C}^*$-orbit is stable $\Leftrightarrow$ Mumford weight $> 0$.

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Space of Kähler forms ((Calabi style)

$\xrightarrow[\text{Moser}]{}$ space of almost complex strs.

$\xrightarrow[\text{Hilbert-Mumford criterion}]{}$ "test configuration" "DF inv"

<u>Def</u> A pair of $(M, L)$ of compact complex manifold $M$ and an ample line bdle $L$ on $M$ is called a polarized manifold.

$c_1(L) > 0$

$$\Omega = \{ \underbrace{\omega}_{\text{Kähler form}} \mid [\omega] \in c_1(L) \}$$

space of Kähler forms.

- ~~K~~Kodaira embedding

$$\iota : M \hookrightarrow \mathbb{P}^N(\mathbb{C}) \quad \swarrow \mathcal{O}(1)$$

$$L = \iota^* \mathcal{O}(1).$$

- Kodaira vanishing

If $L > 0$ then $\exists n_0$ s.t.

for $\forall n \geq n_0$, $H^i(M, L^n) = 0$

for $\forall i > 0$.

(So only $H^0(M, L^n)$ remain)

Rieman-Roch $\longrightarrow$ $\dim H^0(M, L^k)$

$$= \frac{c_1(L^k)^m}{m!} + \cdots$$

polynomial on $k$ of degree $m$.