Lecture No 23   May 27, 2022 (Fri)

§26 Stochastic PDE approach to random interfaces

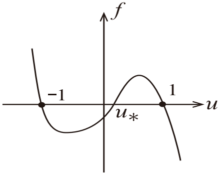
Finally, we discuss a specific problem motivated by physics.

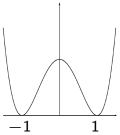
- We consider stochastic Allen-Cahn equation (=TDGL equation, Dynamic $P(\phi)$-model) for $u = u^\varepsilon(t, x, \omega)$:

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u) + \dot{W}^\varepsilon(t, x),$$

- Here $\dot{W}^\varepsilon(t, x)$ is a space-time noise depending on a small parameter $\varepsilon > 0$. (Space-time Gaussian white noise only in 1D)
- Reaction term $f : \mathbb{R} \to \mathbb{R}$ is bistable and balanced:

$$\int_{-1}^{1} f(u)du = 0 \quad \text{or equivalently} \quad V(1) = V(-1).$$

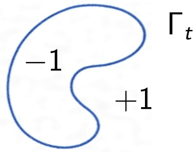




- One can expect that an interface Γ_t appears such that

$$u^\varepsilon(t,x) \underset{\varepsilon\downarrow 0}{\longrightarrow} \chi_{\Gamma_t}(x) := \begin{cases} +1, & \text{outside of } \Gamma_t, \\ -1, & \text{inside of } \Gamma_t, \end{cases}$$

- Problem: Determine the time evolution of Γ_t.
  (Sharp interface limit $\to$ Part C)



- Γ_t would move randomly and the evolution would be governed by some SPDE. Study such SPDEs. ($\to$ Part B)

Part A: Background and Preliminary

1. Introduction
   1.1. Drumhead model
   1.2. TDGL equation (Stochastic Allen-Cahn equation, Dynamic $P(\phi)$-model)
2. Semilinear stochastic PDEs of parabolic type
   2.1. Concepts of Solutions
   2.2. Regularity of Solutions
   2.3. Invariant measures, reversible measures (infinite-dimensional case)

Part B: Stochastic motion by mean curvature

3.1. Background
- 3.1.1. Motion by mean curvature (MMC without noise)
- 3.1.2. Its derivation under sharp interface limit (SIL)
- 3.1.3. Stochastic MMC (SMMC)

3.2. A quick survey of known results
- 3.2.1. Motion by mean curvature
- 3.2.2. Stochastic MMC

3.3. Some further progress
- 3.3.1. SMMC with a direction-dependent smooth noise (DFY)
- 3.3.2. Volume preserving MMC with noise (FY)

Part C: Sharp interface limit

4.1. Sharp interface limit (SIL) without noise

4.2. Sharp interface limit with noise

- 4.2.1. $d = 1$
- 4.2.2. $d \geq 2$
- 4.2.3. Stochastic mass-conserving Allen-Cahn equation
- 4.2.4. The case with boundary condition (Lee)

Part A: Background and Preliminary

1. Introduction

1.1. Drumhead model

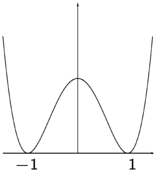
- Kawasaki (2001 Boltzmann medalist)

(from web of Duke Univ)

- Hamiltonian called Ginzburg-Landau-Wilson free energy of the field $u : \mathbb{R}^d \to \mathbb{R}$:

$$H(u) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2} |\nabla u(x)|^2 + V(u(x)) \right\} dx,$$

- $V : \mathbb{R} \to \mathbb{R}$: self-potential of double well type with two bottoms ± 1 of the same level: $V(1) = V(-1)$.

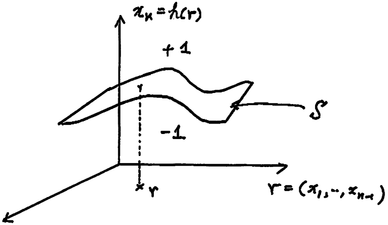


- As the depth of the potential V becomes steep, we expect $u(x) \sim \pm 1$ for each $x \in \mathbb{R}^d$.
- A surface (hypersurface) called an interface S appears in $\mathbb{R}^n$ and separates two different phases $\{\pm 1\}$.

- Assume S is described as a graph:

$$S = \{x = (r, x_n); x_n = h(r)\}, \quad r = (x_1, \dots, x_{n-1}),$$

with a height function $h = h(r)$.



- From $H(u)$, by reduction in the normal direction to S, one can derive the effective Hamiltonian for S:

$$F_{DH}(h) = \frac{\sigma^2}{2} \int_{\mathbb{R}^{n-1}} \sqrt{1 + |\nabla_r h|^2}\, dr \left(= \text{surface area of } S \times \frac{\sigma^2}{2}\right),$$

where $\sigma^2 = \int_{\mathbb{R}} m'(z)^2 dz$ is called the surface tension.
- $m = m(z), z \in \mathbb{R}$ is the minimizer of $H(u)$ considered with $d = 1$: $\frac{\delta H}{\delta u(z)}(m) = 0$ such that $m(\pm\infty) = \pm 1$. It is called the standing wave and will be explained later.

Reduction from H to F_{DH}:

- For the derivation of $F_{DH}(h)$, it is essential to observe that the transition of $u(x)$ across S (i.e., to the normal direction $\vec{n}$ to S) behaves as $m = m(z), z \in \mathbb{R}$.
- The normal vector $\vec{n}$ to the surface S at the point (r, x_n) is given by

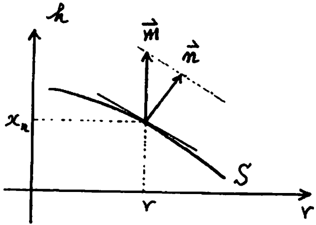
$$\vec{n} = \frac{1}{\sqrt{1 + |\nabla_r h(r)|^2}} \begin{pmatrix} -\nabla_r h(r) \\ 1 \end{pmatrix}$$

$\because$) $\vec{n} \perp \begin{pmatrix} e_i \\ \partial_{x_i} h(r) \end{pmatrix}$ (=tangent vectors to S) and $|\vec{n}| = 1$.

- The change of the interface to the direction $\vec{n}$ is equivalent to the change of the height function h to the vertical direction $\vec{m}$, where

$$\vec{m} = \left(\begin{array}{c} 0 \\ \sqrt{1 + |\nabla_r h(r)|^2} \end{array} \right).$$

$\because$) Check $(\vec{m} - \vec{n}) \perp \vec{n}$.



- Thus, one can expect

$$u(x) = u(r, x_n) \sim m\left(\frac{x_n - h(r)}{a(r)}\right),$$

with $a(r) = \sqrt{1 + |\nabla_r h|^2}$ which describes the width of the layer viewed to the direction $\vec{m}$ instead of $\vec{n}$.
- Insert this into $H(u)$ and we obtain $F_{DH}(h)$. ($\rightarrow$ see next page)
- This is rigorously shown in Γ-convergence.

[Computation leading to $F_{DH}(h)$ from $H(u)$]

Regarding

$$\nabla_{x_i} m\left(\frac{x_n - h(r)}{a(r)}\right) \sim m'\left(\frac{x_n - h(r)}{a(r)}\right)\frac{\nabla_{x_i} h}{a(r)}$$

(by not differentiating $a(r)$ in x_i), and

$$\nabla_{x_n} m\left(\frac{x_n - h(r)}{a(r)}\right) \sim \frac{1}{a(r)} m'\left(\frac{x_n - h(r)}{a(r)}\right),$$

we have

$$H(u) \sim \int_{\mathbb{R}^{d-1}} dr \int_{\mathbb{R}} \left\{ \frac{1}{2} m'\left(\frac{x_n}{a(r)}\right)^2 \frac{1 + |\nabla_r h|^2}{a(r)^2} + V\left(m\left(\frac{x_n}{a(r)}\right)\right)\right\} dx_n$$

by shifting x_n by $h(r)$. Now by the definition of $a(r)$, we have

$$\int_{\mathbb{R}} \left\{ \frac{1}{2} m'\left(\frac{x_n}{a}\right)^2 + V\left(m\left(\frac{x_n}{a}\right)\right)\right\} dx_n = a \int_{\mathbb{R}} \left\{ \frac{1}{2} m'(z)^2 + V(m(z))\right\} dx_n = \frac{a}{2}\sigma^2,$$

where $\sigma^2 = \int_{\mathbb{R}} m'(z)^2 dz$ is called the surface tension; note that $\int_{\mathbb{R}} V(m(z)) dz = 0$ at least if V is symmetric, which we assume. Therefore, one can derive the effective Hamiltonian for the surface S:

$$F_{DH}(h) = \frac{\sigma^2}{2} \int_{\mathbb{R}^{n-1}} \sqrt{1 + |\nabla_r h|^2}\, dr. \qquad \square$$

Relation of $F_{DH}(h)$ to mean curvature

- Note that

$$\frac{\delta F_{DH}}{\delta h(r)}(h) \underset{(1)}{=} -\frac{\sigma^2}{2}\,\mathrm{div}_r\left(\frac{\nabla_r h}{\sqrt{1+|\nabla_r h|^2}}\right) \equiv -\frac{\sigma^2}{2}\kappa(r),$$

  where $\kappa(r)$ denotes the mean curvature of S at (r, x_n) times $(d-1)$ $\quad$ ($\to$ see below)

  [$\because$ for (1)] Take any test function φ and compute

$$\frac{d}{d\varepsilon}\int_{\mathbb{R}^{n-1}}\sqrt{1+|\nabla_r(h+\varepsilon\varphi)|^2}\,dr\bigg|_{\varepsilon=0} = \int_{\mathbb{R}^{n-1}}\frac{\nabla_r h\cdot\nabla_r\varphi}{\sqrt{1+|\nabla_r h|^2}}\,dr$$

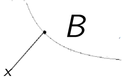
$$= -\int_{\mathbb{R}^{n-1}}\mathrm{div}\left(\frac{\nabla_r h}{\sqrt{1+|\nabla_r h|^2}}\right)\cdot\varphi\,dr.$$

- Kawasaki and Ohta discussed the corresponding dynamic theory.

Mean curvature κ:

- For $B \subset \mathbb{R}^d$, define the distance function from $x \in \mathbb{R}^d$ by

$$\mathrm{dist}(x, B) = \inf_{y \in B} |x - y|.$$



- For $(d-1)$-dimensional hypersurface S given as $S = \partial B$, signed distance function is defined by

$$d(x, S) = \mathrm{dist}(x, B) - \mathrm{dist}(x, B^c), \quad x \in \mathbb{R}^d.$$

- Note: $d(x)$ and d (dimension) should be distinguished.
- It is known that, if S is smooth and d is also smooth in a neighborhood U of S, then the eikonal equation

$$|\nabla d(x)| = 1$$

holds for $x \in U$. In particular, $\nabla d(x)$ is a unit normal vector at $x \in S$ pointing toward the outside of B.

- For $(d-1)$-dimensional hypersurface S, the mean curvature at $x \in S$ is defined as the average of the principal curvatures:

$$\kappa(x) = \frac{1}{d-1} \sum_{i=1}^{d-1} \kappa_i(x), \quad x \in S.$$

- It is known that $\{\kappa_i(x)\}_{i=1}^{d-1}$ and 0 are eigenvalues of the Hesse matrix of d:

$$D^2 d(x) \equiv \mathrm{Hess}\, d(x) = \left(\frac{\partial^2 d}{\partial x_i \partial x_j} \right)_{1 \le i,j \le d}.$$

- Note that, differentiating $|\nabla d(x)|^2 = 1$ in x, we have $(D^2 d)\nabla d = 0$ so that ∇d is the 0-eigenvector of $D^2 d$.
- In particular, we see that

$$\Delta d(x) \equiv \mathrm{Tr} D^2 d(x) = (d-1)\kappa(x).$$

Another ways to express κ:

- If the hypersurface S is represented as a zero set of a C^∞-function u on $\mathbb{R}^d$, that is $S = \{x; u(x) = 0\}$, then we have

$$\nabla d(x) = \frac{\nabla u}{|\nabla u|}(x) \quad \text{on} \;\; S.$$

- Therefore, the mean curvature (times $d-1$) is represented as

$$\Delta d(x) = \text{div}\ \nabla d(x) = \text{div}\ \left(\frac{\nabla u}{|\nabla u|}\right).$$

- If S is represented as a graph $S = \{(r, x_n); x_n = h(r)\}$, one can take $u(x) = h(r) - x_n$ so that $\nabla u = (\nabla_r h, -1)$ and we obtain the formula we stated above:

$$\Delta d(x) = \text{div}_x \left(\frac{(\nabla_r h, -1)}{\sqrt{1 + |\nabla_r h|^2}}\right)$$
$$= \text{div}_r \left(\frac{\nabla_r h}{\sqrt{1 + |\nabla_r h|^2}}\right).$$

1.2. TDGL equation

- Time-dependent Ginzburg-Landau (TDGL) equation
  (cf. Hohenberg-Halperin, Kawasaki-Ohta, ∞-dim Langevin eq)

$$\partial_t u = -\frac{1}{2}(-\Delta)^{\alpha} \frac{\delta H}{\delta u(x)}(u) + (-\Delta)^{\alpha/2} \dot{W}(t, x, \omega), \ x \in \mathbb{R}^d,$$

$\dot{W}(t, x, \omega) =$ space-time Gaussian white noise with mean 0 and covariance structure formally given by

$$E[\dot{W}(t, x)\dot{W}(s, y)] = \delta(t-s)\delta(x-y), \tag{1}$$

$$H(u) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2}|\nabla u(x)|^2 + V(u(x)) \right\} dx.$$

- $\alpha = 0$: Model A (non-conservative system),
  $\alpha = 1$: Model B (conservative system).

- Heuristically, Gibbs measure $\frac{1}{Z}e^{-H}du$ is invariant (reversible) under these dynamics, where $du = \prod_{x\in\mathbb{R}^d} du(x)$ is the Feynman measure.
- Recall fluctuation-dissipation theorem, distorted Brownian motion discussed in Lect-19 in a finite-dimensional setting.

- The functional derivative of $H(u)$ is given by

$$\frac{\delta H}{\delta u(x)}(u) = -\Delta u + V'(u(x)).$$

In fact, for every test function $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned}
&\frac{d}{d\varepsilon} H(u+\varepsilon\varphi)\Big|_{\varepsilon=0} \\
&\quad = \int_{\mathbb{R}^d} \frac{d}{d\varepsilon}\left\{\frac{1}{2}|\nabla u(x) + \varepsilon\nabla\varphi(x)|^2 + V(u(x)+\varepsilon\varphi(x))\right\}\Big|_{\varepsilon=0} dx \\
&\quad = \int_{\mathbb{R}^d} \left\{\nabla u(x)\cdot\nabla\varphi(x) + V'(u(x))\varphi(x)\right\} dx \\
&\quad = \int_{\mathbb{R}^d} \left\{-\Delta u(x) + V'(u(x))\right\}\varphi(x) dx.
\end{aligned}$$

- Therefore, TDGL eq of non-conservative type has the form:

$$\partial_t u = \frac{1}{2}\Delta u - \frac{1}{2}V'(u) + \dot{W}(t,x), \tag{2}$$

while TDGL eq of conservative type has the form:

$$\partial_t u = -\frac{1}{2}\Delta^2 u + \frac{1}{2}\Delta\{V'(u)\} + \sqrt{-\Delta}\dot{W}(t,x). \tag{3}$$

- The noise $\sqrt{-\Delta}\dot{W}$ can be interpreted as the time derivative of a Q-cylindrical Brownian motion on $L^2(\mathbb{R}^d, dx)$ with a covariance operator $Q = -\Delta$.

- Or dropping $\frac{1}{2}$ and writing f for $-V'$, we have
- TDGL eq of non-conservative type (stochastic reaction-diffusion equation, stochastic Allen-Cahn equation, dynamic $P(\phi)$-model) has the form:

$$\partial_t u = \Delta u + f(u) + \dot{W}(t,x), \tag{4}$$

- TDGL eq of conservative type (stochastic Cahn-Hilliard equation) has the form:

$$\partial_t u = -\Delta^2 u - \Delta\{f(u)\} + \nabla \cdot \dot{\mathbb{W}}, \tag{5}$$

where $\dot{\mathbb{W}} = (\dot{W}^i(t,x))_{i=1}^d$ is $\mathbb{R}^d$-valued space-time Gaussian white noise.
- Note that the covariance structure of $\nabla \cdot \mathbb{W}$ is the same as that of $\sqrt{-\Delta}W$:

$$E[\langle \nabla \cdot \mathbb{W}, \varphi\rangle^2] = E\Big[\sum_{i=1}^d \langle W^i, \partial_{x_i}\varphi\rangle^2\Big] = t|\nabla\varphi|^2 = t(-\Delta\varphi,\varphi)$$
$$= t(Q\varphi,\varphi) = t(\sqrt{-\Delta}\varphi, \sqrt{-\Delta}\varphi) = E[\langle \sqrt{-\Delta}W, \varphi\rangle^2].$$

Stochastic PDEs used in physics are sometimes ill-posed.

For example for (2),

- Noise is very irregular: $\dot{W} \in C^{-\frac{d+1}{2}-} := \cap_{\delta>0} C^{-\frac{d+1}{2}-\delta}$ a.s. (Or $\dot{W} \in H_{\text{loc}}^{-\frac{d+1}{2}-} := \cap_{\delta>0} H_{\text{loc}}^{-\frac{d+1}{2}-\delta}$ a.s.)
- Linear case (without $V'(u)$, Schauder estimate):
$$u(t,x) \in C^{\frac{2-d}{4}-,\frac{2-d}{2}-} \text{ a.s.}$$
- Well-posed only when $d = 1$.

Similar for (3):

- Linear case: $u(t,x) \in C^{\frac{2-d}{8}-,\frac{2-d}{2}-}$ a.s.
- Well-posed only when $d = 1$.

See Section 2 for details. (On $\mathbb{T}^d$, we discussed before.)

Theory for ill-posed SPDEs:

- Hairer: Regularity structures, systematic renormalization
- TDGL equation with $V(u) = \frac{1}{4}u^4$:
  =Stochastic quantization (Dynamic $P(\phi)_d$-model):

$$\partial_t \phi = \Delta \phi - \phi^3 + \dot{W}(t,x), \quad x \in \mathbb{R}^d$$

- For $d = 2$ or 3, replace $\dot{W}$ by a smeared noise $\dot{W}^\varepsilon$ and introduce a renormalization factor $-C_\varepsilon \phi$. Then, the limit of $\phi = \phi^\varepsilon$ as $\varepsilon \downarrow 0$ exists (locally in time). Real eq is:
$$\partial_t \phi = \Delta \phi - \phi^3 - \infty \cdot \phi + \dot{W}(t,x)$$
- The solution is continuous in ξ (in place of $\dot{W}^\varepsilon$) and ξ's (finitely many) polynomials. (cf. Rough path theory).
- Global well-posedness: Weber-Mourrat, Hoshino ($\mathbb{C}$-valued case), method of energy inequality
- Another approach by Gubinelli, Imkeller and Perkowski:
  - Paracontrolled calculus (harmonic analytic method)

One way to have better regularity: Non-nearest neighbor interactions

- Replace the Hamiltonian by

$$H(u) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2} \mathcal{A}u(x) \cdot u(x) + V(u(x)) \right\} dx,$$

where $\mathcal{A}$ is a higher order elliptic differential operator:

$$\mathcal{A}u(x) = \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} D^\alpha \{ a_{\alpha\beta} D^\beta u \}(x),$$

and $a_{\alpha\beta} = a_{\beta\alpha}$, positive definite. Originally $\mathcal{A} = -\Delta$, but we take $\mathcal{A} = -(-\Delta)^m$ for example. We have

$$\frac{\delta H}{\delta u(x)} = \mathcal{A}u + V'(u(x)),$$

- Corresponding TDGL equation has a solution with better regularity, see Section 2. (On $\mathbb{T}^d$, we discussed before.)

Relation to Allen-Cahn/Cahn-Hilliard equations

- When $\dot{W} = 0$ (i.e., no noise) and $V=$ double-well type, non-conservative TDGL eq (2) is known as Allen-Cahn equation or reaction-diffusion equation of bistable type, whereas conservative TDGL eq (3) is known as Cahn-Hilliard equation.

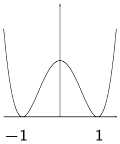
- We will discuss sharp interface limit for Allen-Cahn equation with noise (Stochastic Allen-Cahn equation).

- Dynamic phase transition, Sharp interface limit as $\varepsilon \downarrow 0$ for TDGL equation (=stochastic Allen-Cahn equation):

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u) + \dot{W}(t,x), \quad x \in \mathbb{R}^d \tag{6}$$

$f = -V'$, Potential V is of double-well type:

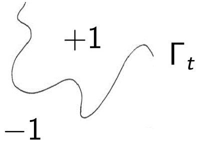
e.g., $f = u - u^3$ if $V = \frac{1}{4}u^4 - \frac{1}{2}u^2$



$\dot{W}(t,x)$ should also be properly scaled.

- The limit is expected to satisfy:

$$u(t,x) \xrightarrow[\varepsilon\downarrow 0]{} \begin{cases} +1 \\ -1 \end{cases}$$



- A random phase separating hyperplane Γ_t appears and the goal is to determine its dynamics under proper time scaling.
- In the limit, we expect to have something like

$$V = \kappa + \dot{W},$$

where V is (inward) normal velocity of Γ_t. κ is mean curvature ($\times(d-1)$) and $\dot{W}$ is certain noise.

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