[KM98] Kollár - Mori's book
[KMM] Kawamata-Matsuda-Matsuki
[Hartshorne] Algebraic Geometry.

I. Introduction

Alg Geometry. Goal: To classify varieties

Birational Geo: To classify varieties up to birational equi

$$\begin{array}{ccc} X & \overset{\text{bir equi}}{\dashrightarrow} & Y \\ \text{Zariski open } \cup\!\! | & & \cup\!\! | \\ \emptyset \neq U & \simeq & V \end{array}$$

MMP: Given ver X, we perform a sequence birational operations $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_k$ such that $X_k = Y$ is "minimal". Then classify Y

I.1 Varieties.

We work over $\mathbb{C}$

Diff Geo viewpoint. $\mathbb{CP}^N = \dfrac{\mathbb{C}^{N+1} - \{0\}}{\mathbb{C}^*}$

where $\mathbb{C}^*$ act on $\mathbb{C}^{N+1}$ by

$$\lambda \cdot (x_0, \cdots, x_N) = (\lambda x_0, \cdots, \lambda x_N)$$

$\mathbb{CP}^N$ is a smooth holomorphic variety.

$$\mathbb{CP}^N \cong \mathbb{C}^N \sqcup \mathbb{CP}^{N-1}$$

It is a natural compactification of $\mathbb{C}^N$

Every point in $\mathbb{CP}^N$ has homogeneous coordinates

$$[x_0 : x_1 : \cdots : x_N]$$

so that $[x_0 : x_1 \cdots : x_N] = [\lambda x_0 : \cdots : \lambda x_N]$

Note: $x_0, \cdots, x_N$ cannot be all zero

<u>Proj Var</u> Def: A proj var X is a subvariety of $\mathbb{CP}^N$ defined as the common zero of some homogeneous polynomials $P_1 \cdots P_k \in \mathbb{C}[x_0, \cdots, x_N]$

Note: There is a diff geo def for subvar. However, Chow's theorem states that every closed analytic subvar is defined globally by homogeneous poly.

Zariski Topology: There is a topo on $\mathbb{C}P^N$ s.t. every closed is a subvariety defined as before. Now, if $X \subseteq \mathbb{C}P^N$ is a proj var, X has an induced Zariski topo. Moreover, it is independent of $X \subseteq \mathbb{C}P^N$.

Quasi-proj: X is a quasi-proj var is a Zariski open in a proj var $\overline{X}$.

Alg Geo Viewpoint: Every proj $X = \{P_1 = \cdots = P_k = 0\}$ gives a scheme of finite type $(X, \mathcal{O}_X)$ over $\mathbb{C}$.

Remark: ① $X(\mathbb{C})$ of the scheme X is just the differential var X.

② $(X, \mathcal{O}_X)$ contains more information than $X(\mathbb{C})$. (It counts multiplicity)

A scheme $(X, \mathcal{O}_X)$ over $\mathbb{C}$ is a variety if it is integral of finite type (Hartshorne)

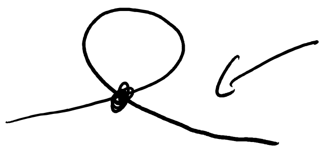
From now on, evey variety is a variety $(X, \mathcal{O}_X)$ as scheme.

Most of the time, we will ommit $\mathcal{O}_X$ and just write X for variety.

If X is a variety, there is an open dense Zariski open X_{ns} s.t. X_{ns} consists of smooth points of X. X_{ns} is called the smooth locus of X

ns is for non singular

Example: $\dim X = 1$



outside this point, X is smooth

Note: X is irreducible $\Longleftrightarrow$ X_{ns} is connected.

Note: $U \subseteq X$ Zariski open.

U is dense $\Longleftrightarrow$ $U \neq \emptyset$

Function field $K(X) = \mathbb{C}(X)$.

Def: X is a quasi-proj var. There is a field $K(x)$ (or $\mathbb{C}(x)$) consisting of rational functions on X.

Remark: • for every point $x \in X$, $K(x) = \mathrm{Frac}(\mathcal{O}_{X,x})$

- $\forall g \in K(X)$, $\exists U \subseteq X$ open s.t. $g \in \mathcal{O}_X(U)$.

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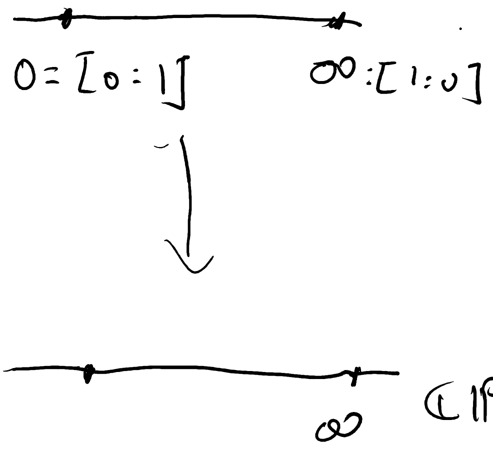
or $g: U(\mathbb{C}) \longrightarrow \mathbb{C}$

is a holomorphic function.

- $K(X)$ "=" {meromorphic functions of X}
- Briefly speaking: a rational function is a regular function defined on some open dense (Hartshorne)

Example: ① $X = \mathbb{P}^1$, every point in $\mathbb{CP}^1$ has coordinate $[x_0 : x_1]$. Then $g: [x_0 : x_1] \mapsto \frac{x_0}{x_1}$ is a rational function

Picture: $X = \mathbb{CP}^1 = \mathbb{C} \sqcup \{\infty\}$.



$\mathbb{CP}^1 = \mathbb{C} \sqcup \{\infty\}$

② $X = \mathbb{P}^2$, $\mathbb{CP}^2 = \{[x_0 : x_1 : x_2]\}$.

$g = \frac{x_0}{x_1}$ is a rational function

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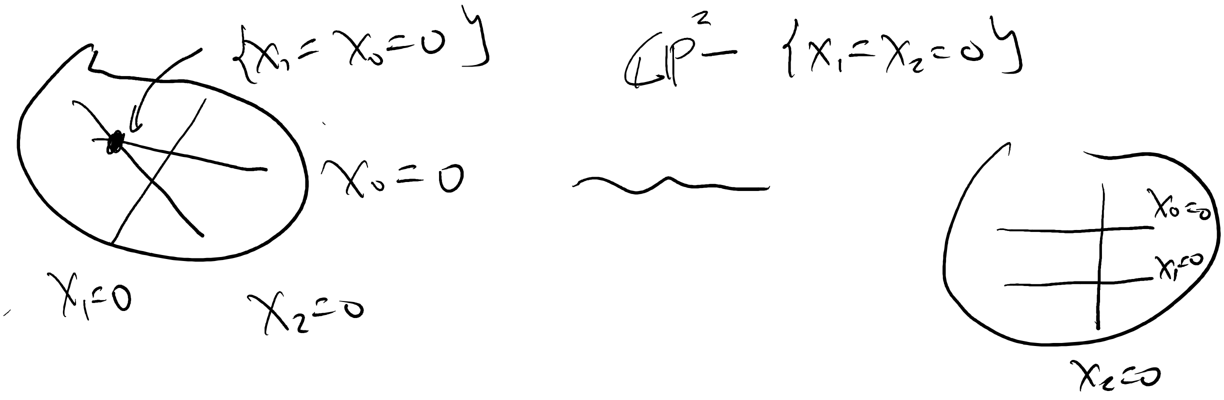
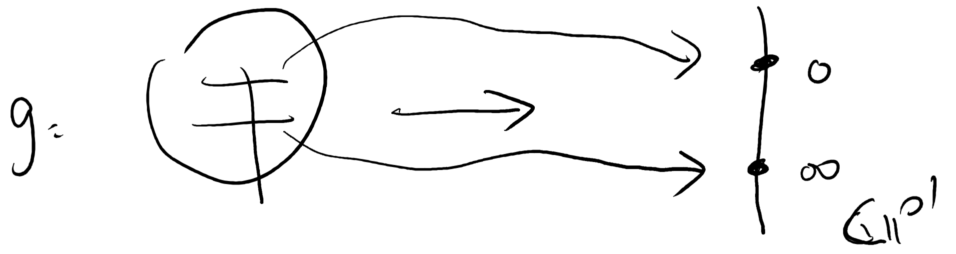




Indeed, in general, X normal, $g \in K(X)$ $g \neq 0$. Then $\exists Z \subseteq X$ closed $\operatorname{codim}_X Z \geq 2$ s.t. g induce a ~~m~~ morphism

$$X - Z \longrightarrow \mathbb{P}^1.$$

(reason: X normal. $\mathbb{P}^1$ is proper)

Hartshorne (seperated and proper morphisms)

Rational and birational Map

Def (Hartshorne) : X, Y 2 varieties. a rational map $f: X \dashrightarrow Y$ is a morphism from a Zariski open dense $U \subseteq X$ to Y ($f|_U: U \to Y$)

Def: $f: X \dashrightarrow Y$ is birational if $\exists U \subseteq X, V \subseteq Y$ Zariski open s.t. $f|_U: U \simeq V$

Remark: If $\overset{X \text{ normal}}{Y}$ is proper, then for $f: X \dashrightarrow Y$ rational, we can choose $U \subseteq X$ so that $\operatorname{codim}(X-U) \geq 2$

Blow-up: $Z \subseteq X$ irreducible closed subvariety $\operatorname{codim} Z \geq 2$. Then we can blowup Z in X, and obtain a morphis $f: X' \longrightarrow X$ which is birational.

This can be viewed as replacing Z by a codim 1 subvariety in X

Hartshorne II.7

Normal varieties: (Hartshorne)

Def: $(X, \mathcal{O}_X)$ is normal, if every $\mathcal{O}_{X,x}$ is integrally closed in $\operatorname{Frac}(\mathcal{O}_{X,x}) = K(X) = \mathbb{C}(X)$

Theorem (KM98, or search online)

X is normal $\iff$

(R1): smooth in codim 1

(S2): Serre condition 2.

A property is satisfied in codim k mean $\exists Z \subseteq X$ closed with $\operatorname{codim} Z \geq k+1$ s.t. $X - Z$ has that property.

Thus R1 means $\exists Z \subseteq X$ closed, $\operatorname{codim} Z \geq 2$ s.t. $X - Z$ is smooth.

(S2) can be view as follows.

for $Z \subseteq X$ closed, $\operatorname{codim} Z \geq 2$

for $U \subseteq X$ open, for $f, g \in \mathcal{O}_U(U)$

If $f|_{U-Z} = g|_{U-Z}$ then $f = g$ (on U)

("Hartog's" Phenomenon)

Remark: If $\dim X = 1$, X normal $\iff$ X smooth

Thm: X any proj variety. There is a unique finite birational

Morphism: $n : X' \longrightarrow X$ s.t. X' is normal

This is called the normalization of X.

Example:

I.2 Divisors (Hartshorne, II.6)

Def: X normal q-proj. a Weil divisor is a linear combination

$$D = \sum_{i=1}^{k} a_i D_i$$ s.t. D_i is an irr. subvariety of codim 1.

A $\mathbb{Q}$- (or $\mathbb{R}$-) weil divisor is an analogue with $\mathbb{Q}$ (or $\mathbb{R}$) coefficients.

The support of $D = \sum_{i=1}^{k} a_i D_i$ is

$$\operatorname{supp} D = \bigcup_{a_i \neq 0} D_i$$ or $$\operatorname{Supp} D = \sum_{a_i \neq 0} D_i$$

Every D_i with $a_i \neq 0$ is called an irreducible component of D.

Prime divisor is $D = D_1$ which means D is irreducible with coeff 1.

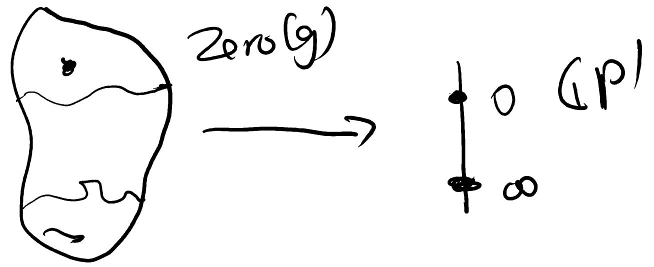
Principal divisor: Consider $g \in K(X)$.

Then $\operatorname{div}(g) = \operatorname{zero}(g) - \operatorname{pole}(g)$.

such a divisor is called a principal divisor.

Interpretation: $g \in K(X)$. $\exists Z \subseteq X$ closed, codim $Z \geq 2$ s.t

$g: X - Z$



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given a Weil divisor D, we can define a sheaf $\mathcal{O}_X(D)$ on X s.t. for $U \subseteq X$ open

$$\mathcal{O}_X(D)(U) = \{ g \in K(X) \mid (\operatorname{div}(g) + D)|_U \geq 0 \}$$

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$\mathcal{O}_X(D)$ is a reflexive sheaf of rank 1.

($\mathcal{F}$ reflexive means $\mathcal{F}^{**} \simeq \mathcal{F}$, $*$ means dual)

Def: D is called a Cartier divisor if $\mathcal{O}_X(D)$ is an invertable sheaf.

Def: Two weil divisors are linearly equi if $D_1 - D_2 = \operatorname{div}(f)$ for some $f \in K(X)$

In this case, we write $D_1 \sim D_2$

Def: $D_1 \sim_{\mathbb{Q}} D_2$ if $a(D_1 - D_2) = \operatorname{div} f$ for some $a \in \mathbb{N}_{>0}$.

Theorem: There are 1-1 correspondences

① $\{\text{weil divisors}\}/\sim \overset{1\text{-to-}1}{\simeq} \{\text{reflexive sheaf of rank } 1\}/\text{iso}$

② $\{\text{Cartier divisors}\}/\sim \overset{1\text{-to-}1}{\simeq} \{\text{invertible sheaf}\}/\text{iso}$

$\overset{\text{1-to-1}}{\cong}$ {line bundle} / iso

Canonical divisor : X normal variety.

Then Ω_X^1 differential sheaf on X.

(Note, over smooth locus, Ω_X^1 is the cotangent bundle of X)

Ω_X^1 has rank $= \dim X$.

$\omega_X = \left(\bigwedge^{\dim X} \Omega_X^1 \right)^{**}$ is a reflexive sheaf of rank 1.

Then there is a Weil divisor K_X s.t.

$\mathcal{O}_X(K_X) \cong \omega_X$. K_X is called a canonical divisor

Note: K_X is not unique, but unique up to linear equivalent

Pull back & Strict Transform

Def: $f: X \to Y$ a morphism, D a Cartier divisor on Y. Then there is a divisor

f^*D on X s.t. $\mathcal{O}_X(f^*D) \simeq f^*(\mathcal{O}_Y(D))$.

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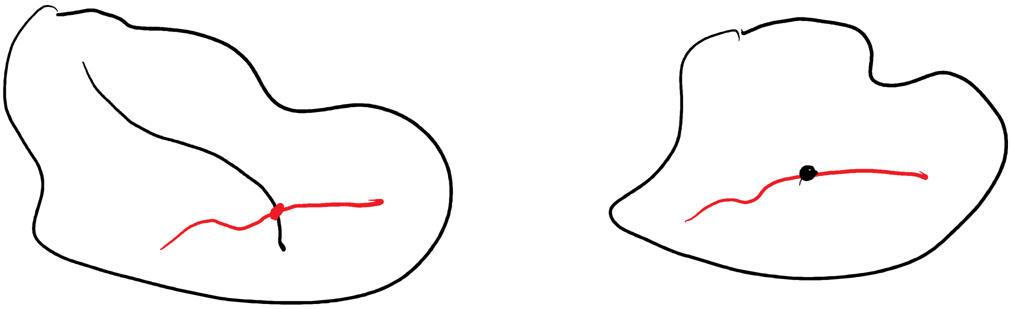
Def Let $f: X \dashrightarrow Y$ a birational map

D a prime divisor on X.

Then the strict transform is

$$f_* D = \begin{cases} f(D) & \text{if } \operatorname{codim} f(D) = 1 \\ 0 & \text{if } \operatorname{codim} f(D) \geq 2. \end{cases}$$

Example



The strict transform of black curve is 0 as it is contracted to a point

Intersection Numbers

X normal q-proj

Let $C \subseteq X$ be a curve (irreducible, proj)

Let D be a Cartier divisor.

Then the intersection number $C \cdot D = \deg_{\tilde{C}} f^* D$

where $\tilde{C} \to C$ is normalization

and $f: \tilde{C} \to C \to X$ is the composition.

$$\begin{tikzcd} \tilde{C} \arrow[r] \arrow[rr, bend right, "f"'] & C \arrow[r] & X \end{tikzcd}$$

Example: ① If $C \cap \text{Support } D = \phi$

then $C \cdot D = 0$

② If D is prime, and $C \cap D \neq \phi$

and $C \not\subseteq D$, then $C \cdot D > 0$

③ if $C \subseteq D$, D prime. then

$C \cdot D$ can negative.

A divisor D is $\mathbb{Q}$-Cartier is

$\exists\, a > 0$, $a \in \mathbb{Z}$ s.t aD is integral

and Cartier.

$\boxed{\text{Next: ample, big divisors, klt sing [KM98, 2]}}$