2020-9-14 Kähler geometry

$\underset{\sim}{y} = A\underset{\sim}{x}$, $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\longrightarrow$

清華園

二校门

column vector

western culture

園華清 eastern culture

$\longleftarrow$

$\underset{\sim}{x} A = \underset{\sim}{y}$, $(x_1 \cdots x_n)\, A = (y_1 \cdots y_n)$

row vector

We employ the western culture, and we express vectors as column vectors.

(Both are possible.)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, $\dim = n$.

$e_1 \ \cdots \ e_n$

Let $e_1, \ldots, e_n$ be a basis.

Then

$$\underline{X = x^1 e_1 + \cdots + x^n e_n}$$

But we want to express a vector as a column vector

$$X = (e_1 \cdots e_n)\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

$$= ex$$

Note that we used upper index for coefficients, and lower index for basis.

This is a standard notation in differential geometry and physics.

Einstein convention

$$X = x^i e_i = e_i x^i$$

omitting $\sum_{i=1}^{n}$ Understand that the sum is taken if upper and lower index appear at the same time.

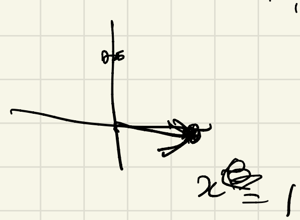
Take the dual basis $e^1, \cdots, e^n$ of V^* w.r.t. $e_1, \cdots, e_n$.

$$\langle e^i, e_j \rangle = \delta^i{}_j \quad \text{Kronecker delta}$$

Take $\alpha \in V^*$ is expressed as

$$\alpha = \alpha_i e^i \left(= \sum_{i=1}^{n} \alpha_i e^i \right).$$

$$x = x^i e_i$$



$f_1 \cdots f_n$

Let $f_1, \cdots, f_n$ be another basis

Then $\exists P$ non-singular matrix s.t.

$$(f_1 \cdots f_n) = (e_1 \cdots e_n) P, \quad P = (p^i{}_j)$$

$$f_j = e_i p^i{}_j = p^i{}_j e_i \quad \left(= \sum p^i{}_j e_i\right)$$

$$P \in \mathrm{End}(V) = V \otimes V^*$$

$$\rotatebox{45}{=}\quad p^i{}_j \; e_i \otimes e^j$$

$$P = \cancel{(p^i{}_j)}$$

$$= (p^i{}_j)$$

tex notation

p^i{}_j

Let $T : V \longrightarrow V$ be a linear map

$$T(e_j) = \sum_{i=1}^{n} a^i{}_j \, e_i$$

$$= e_i \, a^i{}_j \quad \Longleftarrow$$

$$T(e_1 \cdots e_n) = (T(e_1), \cdots, T(e_n))$$

$$= (e_1 \cdots e_n) A \quad \longleftarrow$$

A is called the matrix expression of the linear map T. w.r.t. $e_1, \cdots, e_n$.

$$T(f_1 \cdots f_n) = (T(f_1) \cdots T(f_n))$$

$$= (f_1 \cdots f_n) A'$$

What is the relation of A and A' ?

Answer $A' = P^{-1} A P$.

∵

$$T(f_1 \cdots f_n) = (f_1 \cdots f_n) A'$$

$$\| \qquad = (e_1 \cdots e_n) P \cdot A'$$

$$T((e_1 \cdots e_n) P) = (T(e_1) \cdots T(e_n)) P$$

$$= (e_1 \cdots e_n) AP$$

$$PA' = AP \qquad \therefore \; P = A^{-1} P A'.$$

∴

Following backwards this arguments, we see that

if $A' = P^{-1} A P$ then A and A' are expressions of a same linear map T with respect to different bases.

Vector bundles.

Let M be a smooth manifold of dim n.

Definition

$\pi : E \to M$ between smooth manifold E and M is said to be a real or complex vector bundle of rank r if

(i) $\dim E = n+r$ (or $n+2r$)

(ii) π is a smooth map and the rank of $d\pi$ is n everywhere (maximal rank)

(iii) There is an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of M with diffeo

$$\varphi_\lambda : \pi^{-1}(U_\lambda) \longrightarrow U_\lambda \times \mathbb{R}^r \quad (\text{or } U_\lambda \times \mathbb{C}^r)$$

such that

(iii a) for the projection $p_\lambda : U_\lambda \times \mathbb{R}^n \longrightarrow U_\lambda$

$\pi = p_\lambda \circ \varphi_\lambda$ i.e. local triviality

$$\begin{tikzcd}
E \supset \pi^{-1}(U_\lambda) \arrow[r] \arrow[dr, "\pi"'] & U_\lambda \times \mathbb{R}^r \arrow[d, "p_\lambda"] \\
& p \in U_\lambda \subset M
\end{tikzcd}$$

$$\varphi_\lambda\left(\pi^{-1}(p)\right) = p_\lambda^{-1}(p)$$

(iii b) When $U_\lambda \cap U_\mu \neq \emptyset$

$$\varphi_\lambda \circ \varphi_\mu^{-1} : (U_\lambda \cap U_\mu) \times \mathbb{R}^r \longrightarrow (U_\lambda \cap U_\mu) \times \mathbb{R}^r$$

is expressed as

$$\varphi_\lambda \circ \varphi_\mu^{-1}(p, x) = (p, \varphi_{\lambda\mu}(p)\, x)$$

$$\varphi_{\lambda\mu} : U_\lambda \cap U_\mu \longrightarrow GL(r, \mathbb{R})$$

$\mathbb{C}$

Remark

1. $\{\varphi_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$ are called the transition functions.

2. $\pi^{-1}(p)$ is called a fiber, and has a structure of vector space because

$$\left(\varphi_\lambda\big|_{\pi^{-1}(p)}\right)^{-1} : \underline{p\times\mathbb{R}^r} \longrightarrow \pi^{-1}(p)$$

and the vector space structure $p\times\mathbb{R}^r$ is independent of $\lambda\in\Lambda$. by (iiib)

3. Replacing $\mathbb{R}$ by $\mathbb{C}$ and <u>assuming</u>

   (a) M is a complex manifold.

   (b) E is also a complex manifold.

   (c) all maps are holomorphic

   we say E is a holomorphic vector bundle.

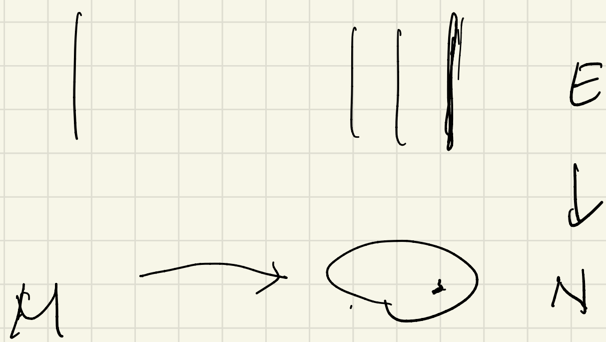
4. Typical examples of vector bundles are tangent bundles and cotangent bundles.

and their tensor products.

5. When $r=1$, E is called a <u>line bundle</u>. often denoted by L.

6. $\pi : E \longrightarrow N$ vector bundle over N, and $f : M \longrightarrow N$ smooth, then we have the pull-back bundle

$$f^*E = \{ (x, v) \in M \times E \mid v \in \pi^{-1}(f(x)) \}$$



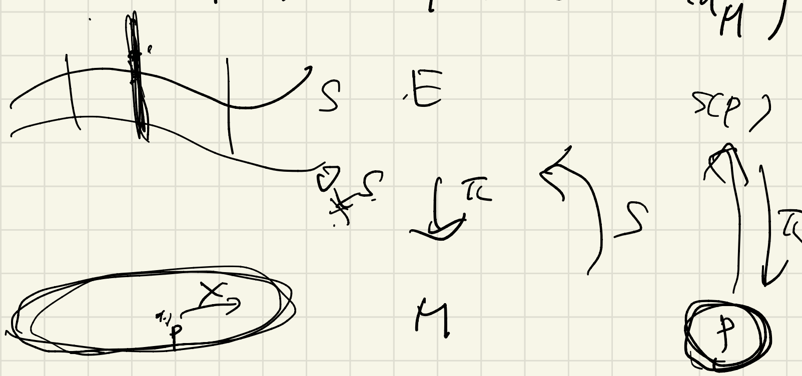
$$M \ni x \longmapsto f(x)$$

7. E is called a trivial bundle if it is a pull-back bundle of the product $\underline{M \times \mathbb{R}^r}$ or $M \times \mathbb{C}^r$

Kähler geometry

$E \xrightarrow{\pi} M$ $\quad C^\infty$ complex vector bundle.

$$C^\infty(E) = \{ s : M \xrightarrow{C^\infty} E \mid \pi \circ s = \mathrm{id}_M \}$$



(M is a smooth mfd.)

$$C^\infty_{\mathbb{C}}(M) = \{ \mathbb{C}\text{-valued smooth functions on } M \}.$$

$f \in C^\infty_{\mathbb{C}}(M)$, $s \in C^\infty(E) \Longrightarrow fs \in C^\infty(E)$.

Def $\nabla : C^\infty(TM \otimes \mathbb{C}) \times C^\infty(E) \to C^\infty(E)$

$$(X, s) \longmapsto \nabla_X s$$

is called a (linear) connection if

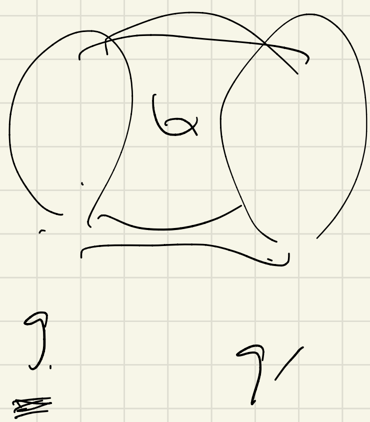
(i) $\underline{\nabla_{fX} s = f \nabla_X s}$ $\qquad f \in C^\infty_c(M)$

(ii) $\underline{\nabla_X (fs) = (Xf)s + f \nabla_X s}$

Def $\nabla_X s$ is called the covariant derivative of s in the direction of X.

Exercise Show using (i) that if $\underline{X_p = Y_p}$ then $\nabla_X s = \nabla_Y s$ at p.

$$\begin{array}{c} E \\ \downarrow \\ M \end{array}$$



Def $e_1, \dots, e_r \in C^\infty(E|_U)$ $\pi^{-1}(U)$

U is an open set, is called a local frame field if

$e_1(p), \dots, e_r(p)$ form a basis of E_p for $\forall p \in U$.

$\mathbb{C}^r$

$E|_U \cong U \times \mathbb{R}^r$

$e_i(p) = (p, e_i)$

Def Given such a frame, define 1-forms $\theta^i{}_j$ by $\cancel{\theta^i_j}$ $\theta^i{}_j$

$$\nabla e_j = \theta^i{}_j e_i \quad \left(= \sum e_i \theta^i{}_j \right)$$

So $$\nabla_X e_j = \sum_{i=1}^{r} \theta^i{}_j(X) e_i.$$

<u>Def</u> $\theta = (\theta^i_j)$ is called the

<u>connection form</u> of ∇ w.r.t. $e_1, \cdots, e_r$.

connection matrix.

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