More on classical limit of the model with hol. pol. of z_0, z_1

$V = P_N(z_0, z_1)$, V is a Hilbert space (Hermitean) with the metric

$$\langle \tilde{P}_N, P_N \rangle = \int d^2 z_0 \, d^2 z_1 \, \overline{\tilde{P}_N} P_N \exp(-|z_0|^2 - |z_1|^2)$$

In this metric $\left(z_i \frac{\partial}{\partial z_j}\right)^+ = z_j \frac{\partial}{\partial z_i}$

4 Herm. operators: $\mathcal{E} = z_1 \frac{\partial}{\partial z_1} + z_0 \frac{\partial}{\partial z_0}$ equals to N

$\frac{1}{2}\left(z_1 \frac{\partial}{\partial z_1} - z_0 \frac{\partial}{\partial z_0}\right) = T_3$, $\frac{1}{2}\left(z_1 \frac{\partial}{\partial z_0} + z_0 \frac{\partial}{\partial z_1}\right) = T_1$

$$T_2 = \frac{i}{2}\left(z_1 \frac{\partial}{\partial z_0} - z_0 \frac{\partial}{\partial z_1}\right)$$

In physics they are called spin (or momentum) operators. One can show

$$T_1^2 + T_2^2 + T_3^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right)$$

In physics $j = \frac{N}{2}$

$T_a^{cl} = \frac{T_a}{j}$, $[T_a^{cl}, T_b^{cl}] = \frac{i}{j} \epsilon_{abc} T_c^{cl}$ (A)

when $j \to \infty$ ($N \to \infty$) the algebra A tends to commutative algebra, with generators X_a, such that $X_1^2 + X_2^2 + X_3^2 = 1$

this is a sphere S^2

---

let us study how points on this S^2 appear

Recall the concept of dispersion of a stoch. observable $\mathcal{O}$:

$$\mathcal{D}(\mathcal{O}) = \langle \mathcal{O}^2 \rangle - (\langle \mathcal{O} \rangle)^2$$

$$\mathcal{D}(\mathcal{O}) \geq 0 \quad \langle (\mathcal{O} - \langle \mathcal{O} \rangle)^2 \rangle \geq 0$$

$$\langle \mathcal{O}^2 \rangle - \cancel{2} \langle \mathcal{O} \rangle \langle \mathcal{O} \rangle + \cancel{\langle \mathcal{O} \rangle^2} \geq 0$$

$\langle\quad\rangle$ - taking average depends in QM on the state, over which we are getting the probability distribution.
The rule is $\langle \mathcal{O}\rangle = Tr(\mathcal{O}\, P_\psi) =$

$$(Av) \quad = \frac{\langle \psi, \mathcal{O}\psi\rangle}{\langle\psi,\psi\rangle}$$ (where vector ψ represents the state $\in P(V)$)

We will look at states of minimal dispersion
__Claim__: These states are of the form:
$\psi_\alpha = (\alpha_0 z_0 + \alpha_1 z_1)^N$, i.e. they correspond to the image of the Veronese map from algebraic geometry $\mathbb{C}P^1 \to \mathbb{C}P^N$.

projectivization of space of pol. of degree N

By minimal dispersion I mean total dispersion of 3 observables, T_1, T_2, T_3, i.e.

$$\langle (T_1^2 + T_2^2 + T_3^2)\rangle - (\langle T_1\rangle)^2 - (\langle T_2\rangle)^2 - (\langle T_3\rangle)^2$$

$$T.D. = \frac{N}{2}\left(\frac{N}{2}+1\right) - \langle T_1\rangle^2 - \langle T_2\rangle^2 - \langle T_3\rangle^2$$

I am looking for maximal sum of squares of averages.
Consider examples: Ex. 1. $N=1$
One may show that each state can be transformed by $SU(2)$ rotation (sym. of the problem) to $\begin{pmatrix}1\\0\end{pmatrix}$, $\langle T_2\rangle_{\binom{1}{0}} = \langle T_1\rangle_{\binom{1}{0}} = 0$

$$\langle T_3\rangle_{\binom{1}{0}} = +\frac{1}{2}\cdot 1 + \left(-\frac{1}{2}\right)\cdot 0 = \frac{1}{2}$$

$$\mathcal{D} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

The only possible dispersion

Ex. 2: $N=2$:

$$\psi_\uparrow = z_1^2 \qquad \langle T_2\rangle_\uparrow = \langle T_3\rangle_\uparrow = 0 \quad \langle T_1\rangle_\uparrow = 1$$

$T.\mathcal{D}_{\uparrow} = \left(\frac{N}{2}\left(\frac{N}{2}+1\right)=3\right) - 1 = 2$

$\Psi_{\uparrow\downarrow} = z_1^2 - z_0^2 \longleftrightarrow$ not of the Veroneze form (not a complete square)

Also

$\langle T_1 \rangle = \langle T_2 \rangle_{\uparrow\downarrow} = 0$

$\langle T_3 \rangle_{\uparrow\downarrow} = (+1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$

(due to z_1^2) probability probability

How did I computed probabilities?

$\mathcal{O} = \sum_i \lambda_i \, Pr_i \;;\quad \langle \mathcal{O} \rangle_\Psi = \sum_i \lambda_i \, \mathrm{Tr}\, Pr_i \, P_\Psi =$

$$= \sum_i \lambda_i \underbrace{\frac{|\langle i, \Psi \rangle|^2}{\langle \Psi, \Psi \rangle}} \longrightarrow \text{probability}$$

$\Psi = \underline{\underline{z_1^2 - z_0^2}}$ $\quad z_1^2$ – eigenvector for eigenvalue 1

z_0^2 – eigenvector for eigenvalue -1

both probabilities are equal and equal to $\frac{1}{2}$.

And Total dispersion now is $3 > 2$.

Let us study large N dispersion of the Veroneze state $(\alpha_0 z_0 + \alpha_1 z_1)^N$

1) By $SU(2)$ rotation we may put it into

$\Psi_{\uparrow N} = (z_1)^N$, then $\langle T_1 \rangle_{\uparrow N} = \langle T_2 \rangle_{\uparrow N} = 0$

Each T_i changes the degree of z_{1N}, so it produces state orthogonal to (z_1),

but $\langle T_3 \rangle_{\uparrow N} = N/2$ (probability is 1)

The $T\mathcal{D}$. of T is $\frac{N}{2}\left(\frac{N}{2}+1\right) - \left(\frac{N}{2}\right)^2 = \frac{N}{2}$.

Now, $T.\mathcal{D}$ of T^{cl} is

T.D. of T^{cl} is $\frac{\text{TD of } T}{\left(\frac{N}{2}\right)^2} = \frac{2}{N}$

TD of $T^{cl} \to 0$ when $N \to \infty$

At the same time

TD of state of the T $z_1^N - z_0^N = \frac{N}{2}\left(\frac{N}{2}+1\right)$

T.D of T^{cl} of this state $= 1 \not\to 0$

clear, since probability to find $T_3^{cl} = +1$ and $T_3^{cl} = -1$ are equal and

such state is far from give a definite answer on the question what are values of almost commuting observables T_a^{cl}.

Given a minimal disp. state MD,

$\langle T_a \rangle = \frac{\langle MD | T_a | MD \rangle}{\langle MD | MD \rangle}$ – it is a vector in $\mathbb{R}^3$

by su(2) rotation I may turn it to be along the third axis (su(2) rotation of $|MD\rangle$ state)

Thus, I am looking for maximal $(\langle T_3 \rangle)^2$

It can be only $\langle T_3 \rangle = \frac{N}{2}$ or $\langle T_3 \rangle = -\frac{N}{2}$

for states z_1^N or z_0^N that belong to Veronese states $\square$.

Veroneze states tend to orthogonal basis in the space of states when $N \to +\infty$.

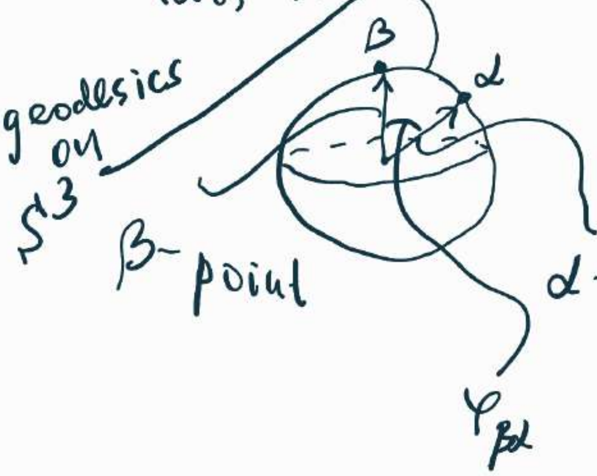
(They do not form such a basis for finite N, really, there are infinitely many of them while the space is finite dim $(N+1)$ for finite N).

Consider two states $(\alpha_0 z_0 + \alpha_1 z_1)^N$ and z_1^N (I can always put the second state

$(\beta_0 z_0 + \beta_1 z_1)^N$ to z_1^N by $SU(2)$ rotation

$\langle z_1^N, (\alpha_0 z_0 + \alpha_1 z_1)^N \rangle = \cos^N \varphi$ | I assume that $|\alpha_0|^2 + |\alpha_1|^2 = 1$

(α_0, α_1) is a point on the 3-dim sphere given by ↑.



taking $\beta_g = 1$.

$\langle \beta | \alpha \rangle = \cos^N \varphi_{\beta\alpha} \to 0$ when $N \to +\infty$.

In the $N \to \infty$ limit Veroneze states behave like points!

Generalization of this example: Hamiltonian reduction

Given a Kahler manifold X with the Kahler form ω consider the holomorphic action of the complex Lie algebra: $\mathfrak{g} \to \mathrm{Vect}\, X$

$t_a \mapsto v_a$, such that compact part of it preserves Kahler form: $(\mathcal{L}_{v_a} - \mathcal{L}_{\bar{v}_a})\,\omega = 0$

In the example we studied we had

$X = \mathbb{C}^2 \quad \mathfrak{g} = \mathbb{C}^* \quad v = z \frac{\partial}{\partial z}, \ \bar{v} = \bar{z} \frac{\partial}{\partial \bar{z}}$

$\omega = dz \wedge d\bar{z}$ $\quad v - \bar{v} = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}$

Then there are Hamiltonians H_a:

$$i(v_a - \bar{v}_a)\,\omega = dH_a$$

In our example, $H = |z_0|^2 + |z_1|^2 + \text{const.}$

We may consider so-called Hamiltonian reduction of $X/\mathcal{Y}$, defined as

$$\{H_a = 0\} / \mathcal{Y}_{compact}$$

In our example we studied

$|z_0|^2 + |z_1|^2 - 1 = 0$ — it was S^3 and we divided by the action of $U(1)$

Choice of the constant(s) could give different results

Orbits of $\mathcal{Y}_{noncompact}$ intersect zero level of Hamiltonians at most at one point

---

Proof: Consider $\mathcal{L}_{V_{nonc}} H = \frac{\partial H}{\partial y^i} v^i + \frac{\partial H}{\partial \bar{y}^i} \bar{v}^i =$

$$= 2\frac{\partial H}{\partial y^i}\frac{\partial H}{\partial y^j} g^{ij} > 0$$

So H is monotonous along the trajectory and thus can intersect the zero level only once.

$$\boxed{\frac{X \setminus \text{bad orbits}}{\mathcal{Y}_{complex}} = \frac{\{H=0\}}{\mathcal{Y}\text{-compact}}}$$

hol. quotient is given by Hamilt. quotient

Most important examples are so-called toric manifolds (generalizations of proj. manifolds)

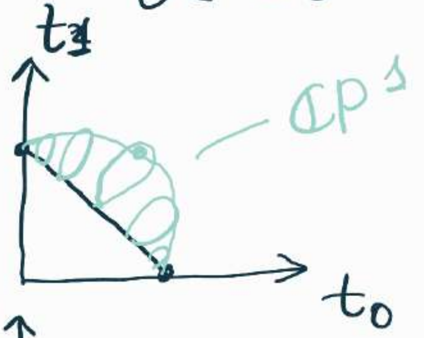
$\mathbb{C}^{N+1}/\mathbb{C}^*$ where $\mathbb{C}^*: \; z_i \to e^{i\varphi} z_i$

$$H = |z_0|^2 + \dots + |z_N|^2 - C$$

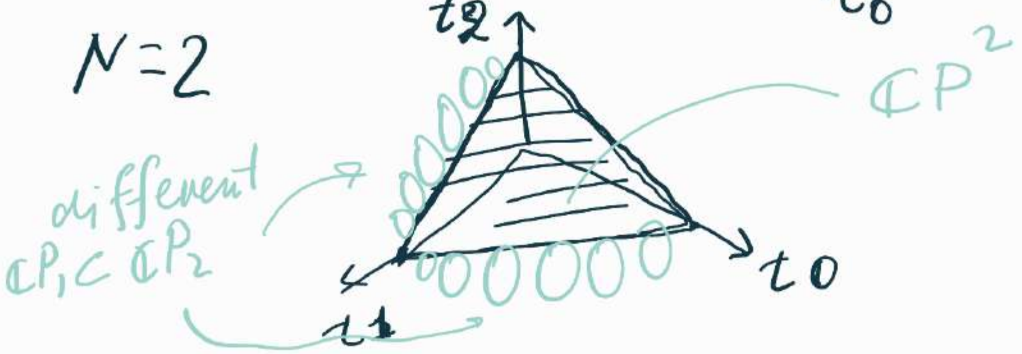
It is conv. to write everything in coord.

$t_i = |z_i|^2$ $\quad 0 \le t_i$

$N = 1$



$N = 2$

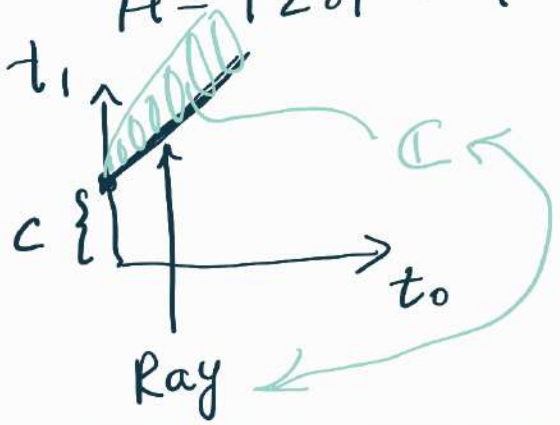


$\mathbb{C}^2$: $\mathbb{C}^*$: $\quad z_0 \to e^{i\varphi} z_0$

$\qquad\qquad\qquad z_1 \to e^{-i\varphi} z_0$

$$H = |z_0|^2 - |z_1|^2 + C$$

$c > 0$

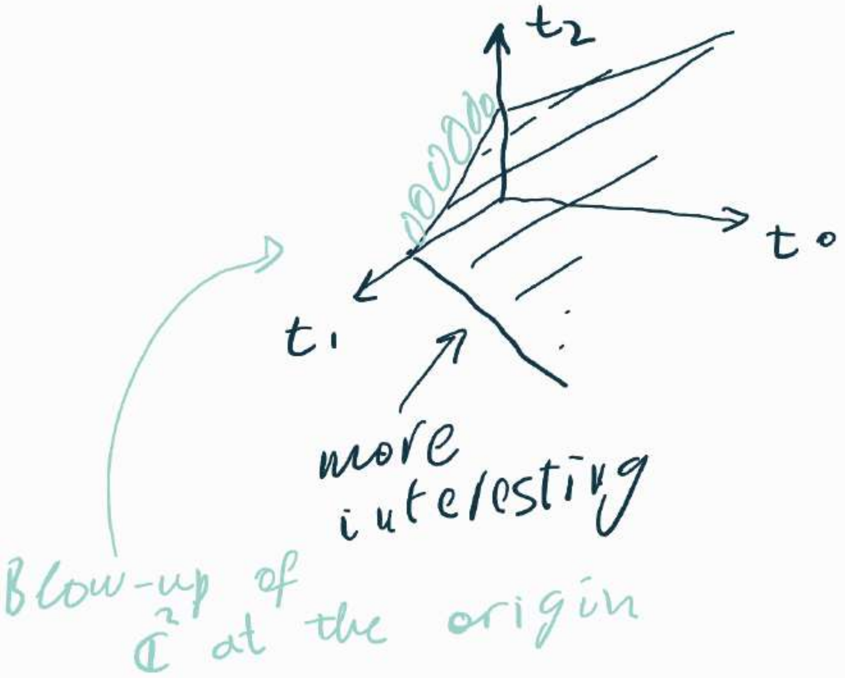


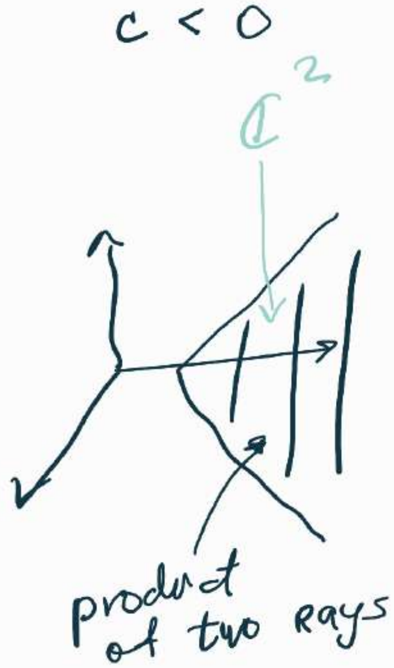
$\mathbb{C}^3$: $\mathbb{C}^*$: $\quad z_0 \to e^{-i\varphi} z_0$

$\qquad\qquad\qquad z_1 \to e^{i\varphi} z_1$

$\qquad\qquad\qquad z_2 \to e^{i\varphi} z_2$

$$H = |z_1|^2 + |z_2|^2 - |z_0|^2 + C$$


t_2
t_0
t_1
more interesting
Blow-up of $\mathbb{C}^2$ at the origin


$c < 0$
$\mathbb{C}^2$
product of two rays