Random hyperbolic and flat surfaces

(Riemann surfaces seen without glasses)

Lecture 5. Benjamini-Schramm Limit

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Motivating example: Permutation.

Let S_n be the group of permutations of the set $\{1, 2, \ldots, n\}$.

A permutation $\sigma \in S_n$ can be understood by its decomposition into cycles.

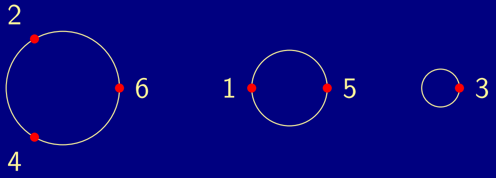
For example, for

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 4 & 1 & 3 \end{pmatrix}$$

we have

$$\sigma = (15)(263)(4).$$

We can visualize σ as a labeled graph X_σ that is a union of cycles.



What does a typical permutation look like?

Question

What is the order of a typical element under a typical permutation $\sigma \in S_n$ as $n \to \infty$?

We equip S_n with a uniform probability measure. Then

$$\begin{aligned}
\mathbb{E}_n(\#\ k\text{–cylces}) &= \frac{1}{n!} \sum_{\sigma \in S_n} (\#\ k\text{–cycles in } \sigma) \\
&= \frac{1}{n!} \# \left\{ (\sigma, c) \,\middle|\, \sigma \in S_n, \quad c \text{ is a cycle in } \sigma \right\} \\
&= \frac{1}{n!} \sum_{\text{cycle c}} (\#\text{ permutations where } c \text{ is a cycle}) \\
&= \frac{\binom{n}{k}(k-1)!(n-k)!}{n!} = \frac{1}{k}
\end{aligned}$$

Benjamini-Schramm limit of a random permutation

For every fixed $k > 0$,

$$\mathbb{E}_n(\#\text{ elements of order at most } k) \leq$$

$$\leq \sum_{j=1}^{k} j \cdot \mathbb{E}_n(\#\ j\text{–cylces}) = \sum_{j=1}^{k} j \cdot \frac{1}{j} = k.$$

Hence, for a random $\sigma \in S_n$, a random $x \in \{1, \dots, n\}$ and $r > 0$,

$$\mathbb{P}_n\Big(\text{order}_\sigma(x) \geq r\Big) \to 1. \qquad \text{as} \qquad n \to \infty.$$

In other words,

$$\mathbb{P}_n\Big(B_r(x) \text{ is a cycle}\Big) \longrightarrow 0 \quad \text{as } n \to \infty.$$

From the point of view of x, X_σ looks like a bi-infinite path

Hausdorff distance

Let (Z, d_Z) be a metric space. For $\epsilon > 0$ and $A \subset Z$, define

$$A^\epsilon = \{z \in Z : \inf_{a \in A} d_Z(a, z) < \epsilon\}.$$

For Borel sets $A, B \subset Z$ define

$$d_Z^H(A, B) = \inf\{\epsilon > 0 : A \subset B^\epsilon \text{ and } B \subset A^\epsilon\}.$$

The **Hausdorff distance** between compact pointed metric spaces (X_1, o_1) and (X_2, o_2) is

$$d_H^c(X_1, X_2) = \inf_{\Phi_1, \Phi_2, Z} \Big(d_Z\big(\Phi_1(o_1), \Phi_2(o_2)\big) + d_Z^H\big(\Phi_1(X_1), \Phi_2(X_2)\big) \Big)$$

where the infimum is over all Polish metric spaces (Z, d_Z) and isometric embeddings $\Phi_i\colon X_i \to Z$, $i = 1, 2$.

Hausdorff Convergence: Regular Polygons to a Circle

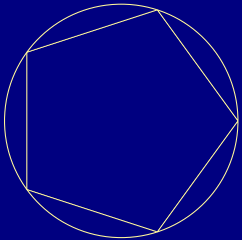
Consider in $\mathbb{R}^2$ the unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \;\big|\; x^2 + y^2 = 1\}.$$

For each integer $n \geq 3$, let P_n be the regular n-gon inscribed in S^1. Endow both P_n and S^1 with the metric induced from $\mathbb{R}^2$.

Then

$$d^H_{\mathbb{R}^2}(P_n, S^1) \to 0 \qquad \text{as} \qquad n \to \infty.$$



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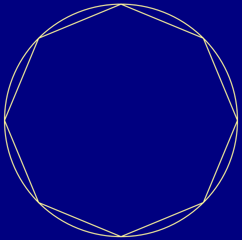

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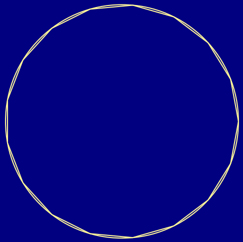


Hausdorff Convergence: A Torus Collapsing to a Circle

In $\mathbb{R}^3$, consider the torus of revolution

$$T_r = \{(x, y, z) \mid (\sqrt{x^2 + y^2} - 1)^2 + z^2 = r^2\}$$

for a small parameter $r > 0$. This is a tube of radius r around the unit circle

$$C = \{(x, y, 0) \mid x^2 + y^2 = 1\}.$$

As $r \to 0$, the sets T_r converge in Hausdorff distance to C. Every point of T_r is within distance r of C, and every point of C is within distance r of T_r.

Topologically, each T_r is a torus, while the limit C is just a circle. The Hausdorff limit forgets the small transverse directions.

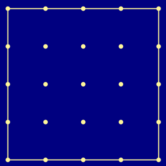
Hausdorff Convergence: Lattice Approximations to a Domain

Let $K = [0,1]^2 \subset \mathbb{R}^2$. For each integer $n \geq 1$, consider the finite set

$$K_n = \frac{1}{n}\mathbb{Z}^2 \cap [0,1]^2,$$

that is, the $(n+1) \times (n+1)$ grid of lattice points in the unit square. Every point of K is within distance at most $\sqrt{2}/n$ of some point of K_n, and every point of K_n lies in K. Hence

$$d^H_{\mathbb{R}^2}(K_n, K) \to 0 \qquad \text{as} \qquad n \to \infty.$$



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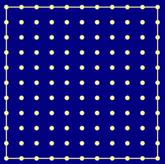

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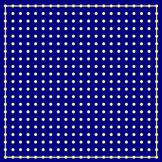


The space of pointed metric spaces

Assuming the ball $X_i^r = B_{X_i}(o_i, r)$ is compact, define

$$d_H(X_1, X_2) = \int_0^\infty e^{-r}\left(1 \wedge d_H^c(X_1(r), X_2(r))\right)\, dr.$$

Let $\mathcal{M}_*$ be the space of pointed-isometry-classes of pointed metric spaces (locally compact length spaces).

Then d_H provides a distance on $\mathcal{M}_*$.

The space $(\mathcal{M}_*, d_H)$ is a Polish metric space.

Polish space: separable completely metrizable topological space.

Length metric space: the distance between two points equals the infimum of the lengths of all paths from one to the other.

Benjamini-Schramm convergence

Let X_n be a sequence of random metric spaces.

$$X_n \quad \xrightarrow{\text{choose a random point}} \quad (X_n, o_n)$$

Let $\mathbb{P}_n$ be the probability measure on $\mathcal{M}_*$ associated to choosing a random point o_n in X_n.

Let $\mathbb{P}$ be another probability measure in $\mathcal{M}_*$. We say X_n Benjamini-Schramm converges to $\mathbb{P}$ if

$$\mathbb{P}_n \xrightarrow{\textit{weakly}} \mathbb{P}$$

When $\mathbb{P}$ is the δ–measure on (X, o) we say

$$X_i \text{ Benjamini-Schramm converges to } (X, o).$$

Benjamini-Schramm convergence: Examples

Let S_R be the sphere of radius R in $\mathbb{R}^3$. Then

$$S_R \text{ Benjamini-Schramm converges to } \mathbb{R}^2.$$

Let Γ_n be random d-regular graph with $2n$ vertices. Then

$$\Gamma_n \text{ Benjamini-Schramm converges to the } d\text{–regular tree.}$$

Let X_g be random hyperbolic surface of genus g. Then

$$X_g \text{ Benjamini-Schramm converges to } \mathbb{H}^2.$$

Random Planar Triangulations

A (rooted) planar triangulation is a finite connected graph embedded in the sphere S^2 such that every face is a triangle, together with a distinguished oriented edge, called the root (up to homeomorphism).

Let $\mathcal{T}_n$ be the set of rooted planar triangulations with n vertices.

We consider a random triangulation T_n chosen uniformly from $\mathcal{T}_n$.

For each integer $R \geq 1$, denote by

$$B_R(T_n, e_n)$$

the ball of radius R around the root e_n.

We ask: as $n \to \infty$, does the law of $B_R(T_n, e_n)$ stabilize, for each fixed radius R?

The Uniform Infinite Planar Triangulation (UIPT)

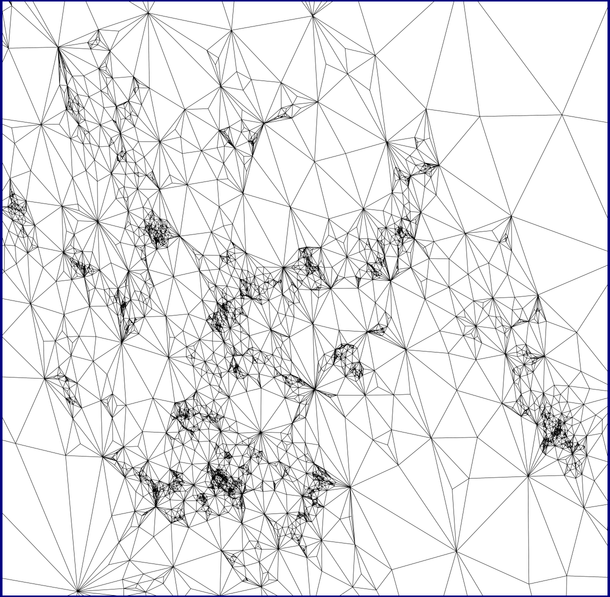
Theorem (Angel–Schramm)

There exists a random rooted infinite planar triangulation (T_∞, o_∞) such that

$$(T_n, o_n) \quad \Longrightarrow \quad (T_\infty, o_\infty)$$

in distribution for the Benjamini-Schramm topology as $n \to \infty$. The limit (T_∞, o_∞) is called the Uniform Infinite Planar Triangulation (UIPT).

The Uniform Infinite Planar Triangulation (UIPT)

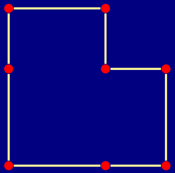


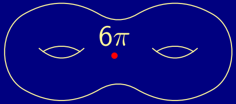
The Space of Translation Surfaces

Let $\mathcal{H}(m_1, \ldots, m_k)$ be the space of translation surfaces of genus g with k singular points $\sigma_1, \ldots, \sigma_k$, where the total angle at σ_i is $2\pi(m_i + 1)$. We have

$$2g - 2 = \sum_{i=1}^{k} m_i$$

For example, the golden L is in $\mathcal{H}(2)$:





$\mathcal{H}_g = \mathcal{H}(1, \ldots, 1)$: the principal stratum

$\mathcal{H}^1(m_1, \ldots, m_k)$: unit area translations surfaces

Masur-Veech Volume

Let (X, ω) be a point in $\mathcal{H}(m_1, \ldots, m_k)$ and Σ be the set of singular points. Let $\gamma_1, \ldots, \gamma_\ell$ be basis for the relative homology. Then

$$(X, \omega) \to \Big(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_\ell} \omega \Big)$$

gives a local coordinate in $\mathbb{C}^\ell$ for a neighborhood of X.

For a set $U \subset \mathcal{H}^1(m_1, \ldots, m_k)$,

$$\mu_{MV}(U) = d \cdot \mu_{\text{Leb}}(\text{Cone}(U))$$

where μ_{Leb} is the Lebesgue measure, d is the dimension and

$$\text{Cone}(U) = [0, 1] \cdot U \subset \mathcal{H}(m_1, \ldots, m_k).$$

The measure μ_{MV} is $\text{SL}(2, \mathbb{R})$ invariant and has finite total measure.

Volume of the Space of Translation Surfaces

$$\mathcal{V}(m_1, \ldots, m_k) = \mu_{MV}\Big(\mathcal{H}^1(m_1, \ldots, m_k)\Big)$$

Theorem (Eskin-Okounkov, Eskin-Zorich, Aggrawal, Chen-Möller-Sauvaget-Zagier)

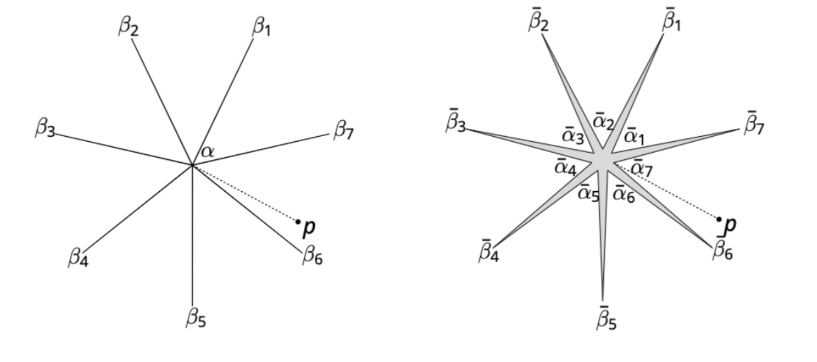
$$\mathcal{V}(m_1, \ldots, m_k) = \frac{4}{\prod_{i=1}^{k}(m_i + 1)} \cdot \Big(1 \pm O\left(1/g\right)\Big)$$

We are interested in the principal stratum $H_g^1(1, \ldots, 1)$. In this case,

$$\mathcal{V}(1, \ldots, 1) \sim \frac{4}{2^{2g-2}} = \frac{1}{2^{2g-4}}$$

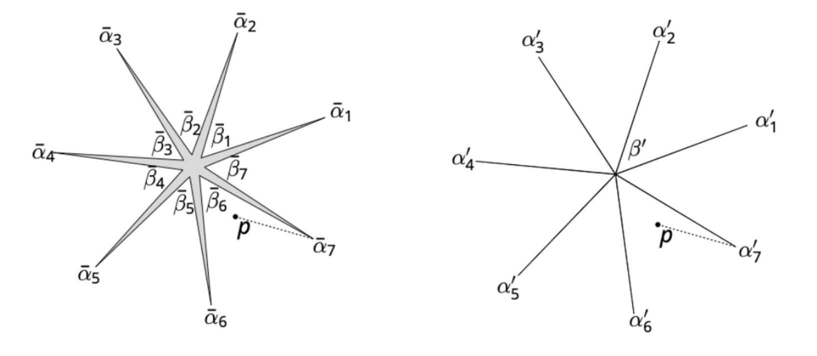
Key tool: star surgery

Equivalent to Schiffer variations, surgeries of Eskin-Masur-Zorich, or the rel flow, star surgery provides geometric perspective on how to moves a singular point around in a translation surface.



Key tool: star surgery

Equivalent to Schiffer variations, surgeries of Eskin-Masur-Zorich, or the rel flow.
Provides geometric perspective so as to keep track of geometry as deformation occurs.



Collapsing Zeros, Eskin-Masur-Zorich

Let $\mathcal{H}_g(\sigma, \tau, r)$ be the set of translation surfaces $(X, \omega) \in \mathcal{H}_g$ where σ and τ are connected by a saddle connection of of length at most r. We have a 2-to-1 volume preserving map

$$\mathcal{H}_g(\sigma, \tau, r) \to \mathcal{H}_g(2, 1, \ldots, 1) \times D_r$$

where $D_r \subset \mathbb{R}^2$ is a disk of radius r.

Therefore,

$$\mathbb{P}_{\mu_{MV}}\big(\mathcal{H}_g(\sigma, \tau, r)\big) = \frac{\mathcal{V}(2, 1, \ldots, 1) \cdot (2\pi r^2)}{\mathcal{V}(1, \ldots, 1)} \asymp r^2.$$

$$\mathbb{P}_{\mu_{MV}}\big(\text{some saddle connection has a length at most } r\ \big) \asymp g^2 \cdot r^2.$$

Central philosophy

Find series of star surgeries to deform to the surface to a more geometrically simple situation

Produce measure-preserving maps between “decorated” strata of translation surfaces

Conclusion:

- No accumulations — iterated collapsing of a shortest path between two singularities
- Description of law of limit — collapsing all singularities near the base point to the base point (constellation collapse)

Straightening Surgery

All simple closed geodesics which pass through a singularity are either closed saddle connections, a pair of homologous saddle connections, or a chain of pair-wise non-homologous saddle connections.

Siegel-Veech theory can handle the first two cases.

Cannot naively collapse the singularities in a chain of saddle connections: you may introduce new saddle connections in the geodesic!

If not careful, your deformation can perpetually increase the length of the simple closed geodesic, or cause it to self-intersect!

Straightening surgery first makes all saddle connections parallel before collapsing all but one or two of them. Tangential self-intersections still occur but can be dealt with.

Steps needed to analyze the limit

We prove the theorem in three conceptual steps.

- The number of zeros in a fixed-radius ball around a generic point is uniformly controlled and does not go to infinity as the genus grows (tightness): the laws of the rooted surfaces form a precompact family for the Benjamini–Schramm topology, so subsequences always admit limiting distributions.
- Ball of radius r is generically a topological disk (Planarity).
- Any two subsequential limits are in fact the same (Ergodicity).

First two steps: existence of subsequential Benjamini–Schramm limits with good local control.

Ergodicity upgrades subsequential convergence to convergence of the whole sequence to a canonical limit.

Benjamini-Schramm Limit of Translation Surfaces

Let μ_X be the measure on (X, ω) associated to the Euclidean metric. Define μ_g^* by

$$d_{\mu_g^*} = d_{\mu_{MV}} \cdot d_{\mu_X}.$$

We now scale a random pointed translation surface chosen according to μ_g^* by a factor of $\sqrt{2g-2}$.
Let $\mathbb{P}_{\mu_g^*}$ denote the associated measure on $\mathcal{M}_*$ normalized to be a probability measure,

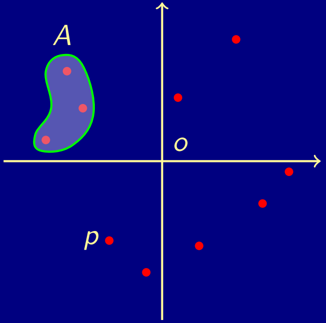
Theorem (Bowen-R-Vallejos)

Every sub-sequential limit of $\{\mathbb{P}_{\mu_g^*}\}_{g=2}^{\infty}$ is the distribution of the Poisson translation plane, with intensity 4.

$$\mathbb{P}_{\mu_g^*} \to \mathbb{P}_{PTP}.$$

Poisson Point Process

A Poisson point process is a process selecting random point in $\mathbb{R}^2$ with the essential feature that the points occur independently of one another.
The rate or **intensity** λ, is the average density of the points in the Poisson process located in some region of space.

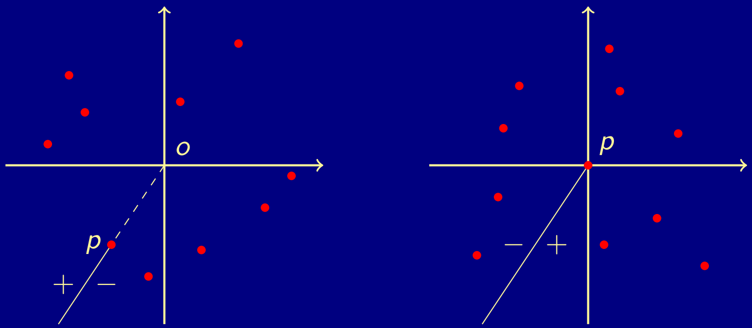


Let R_A be a region of area A.
$N(R_A)$: number of point in R.

$$\mathbb{P}\Big(N(R_A) = n\Big) = \frac{(\lambda A)^n}{n!} e^{-\lambda A}$$

Poisson Translation Plane

Start with a pointed Euclidean plane $(\mathbb{R}^2, o)$.
Attach a copy of $\mathbb{R}^2$ at points given by a Poisson point process with intensity 4.



We denote the resulting probability measure in $\mathcal{M}_*$ by $\mathbb{P}_{PTP}$.
The change of base-point is a groupoid that acts ergodically on $\mathcal{M}_*$ with respect to $\mathbb{P}_{PTP}$.

Collapsing the Visible Zeros

Denote the set of indices of singular points in (X, ω) visible from σ_0 and distance at most r by

$$Z_r = Z_r(X, \omega) \subset \{1, \ldots, n\}$$

For any $T \subset [n] = \{1, \ldots, n\}$, with $|T| = i$ we have

$$\mathbb{P}(T \subset Z) \sim 2^i \cdot \left(2\pi r^2\right)^i .$$

To see this, consider the surgery map (which is $(i+2)$-to-1):.

$$\mathcal{H}_g^1(1, \ldots, 1) \to \mathcal{H}_g^1(i+1, 1, \ldots, 1) \times D_r^i$$

and note that

$$\frac{\mathcal{V}(i+1, 1, \ldots, 1)}{\mathcal{V}(1, \ldots, 1)} \sim \frac{4}{(i+2)2^{n-i}} \Big/ \frac{4}{2^{n+1}} \sim 2^i(i+2).$$

Where Does the Poisson Distribution Come From?

Note that, for any $T \subset [n]$,

$$\mathbb{P}(T \subseteq Z_r) = \sum_{Y \supseteq T} \mathbb{P}(Y = Z_r).$$

Therefore, using inclusion exclusion principal, we have

$$\mathbb{P}(Z_r = T) = \sum_{Y \supseteq T} (-1)^{\# Y - T} \mathbb{P}(Y \subset Z).$$

Hence

$$\begin{aligned} \mathbb{P}\big(|Z_r| = m\big) &= \sum_{|T| = m} \mathbb{P}(Z_r = T) \\ &= \binom{n}{m} \sum_{i=m}^{n} \binom{n-m}{i-m} (-1)^{(i-m)} \cdot 2^i \cdot \left(2\pi r^2\right)^i. \end{aligned}$$

Where Does the Poisson Distribution Come From?

$$\mathbb{P}\left(|Z_r| = m\right) = \binom{n}{m} \sum_{i=m}^{n} \binom{n-m}{i-m} (-1)^{(i-m)} \cdot 2^i \cdot \left(2\pi r^2\right)^i$$

$$\sim \frac{n^m}{m!} \cdot \sum_{i=m}^{n} \frac{n^{(i-m)}}{(i-m)!} (-1)^{(i-m)} \cdot 2^i \cdot \left(2\pi r^2\right)^i$$

Factor out $(4\pi r^2)^m$

$$= \frac{\left(4n\pi r^2\right)^m}{m!} \cdot \sum_{i=m}^{n} (-1)^{(i-m)} \frac{1}{(i-m)!} \cdot \left(4n\pi r^2\right)^{(i-m)}$$

$$= \frac{\left(4n\pi r^2\right)^m}{m!} e^{-4n\pi r^2}.$$

This is the Poisson distribution with $\lambda = 4n\pi r^2$

Scaling by $\sqrt{n}$ is the same as setting $r = \frac{R}{\sqrt{n}}$.

Uniformization of translation surfaces

One can wonder what does a translation surface look like from the point of view of hyperbolic geometry.

For

$$\mathcal{M}_g : \text{moduli space} \qquad \mathcal{ML} : \text{space of measured laminations}$$

there is map

$$\mathcal{H}_1(1,1,\ldots,1) \to \mathcal{M}_g \times \mathcal{ML}_g, \qquad (X,\omega) \to (X,\lambda)$$

The measured lamination λ gives an ideal quadrangulation of X.

Question

What is Benjamini-Schramm limit (X,λ)?

Curien–Werner: Unique Ideal Triangulation of $\mathbb{H}^2$

A random triangulation T of $\mathbb{H}^2$ is assumed to be

- *Möbius-invariant* if its law is invariant under all hyperbolic isometries of $\mathbb{H}^2$.
- *Markovian* ("spatial Markov") if, conditionally on the triangle $T(0)$ that contains the origin, the restrictions of T to the three complementary components of $D \setminus T(0)$ are independent, and the part beyond any side, say (bc), does not depend on the position of the opposite vertex a.

Theorem (Curien–Werner, 2012). There exists a unique in law random complete ideal triangulation T of D that is Möbius-invariant and Markovian in the above sense.

Moreover, they explain how the same construction extends a Möbius-invariant Markovian tiling of $\mathbb{H}^2$ by ideal squares.

Thank You!