Singular Csck metric. Oct 28.

Kähler cone metric ω

- ω is smooth Kähler metric on the regular part $X \setminus D$.
- ω is quasi-isometric to the cone flat metric.

$$\omega_{cone} := \frac{\sqrt{-1}}{2} \beta^2 |z^1|^{2(\beta-1)} dz^1 \wedge d\bar{z}^{\bar{1}} + \sum_{2 \le j \le n} dz^j \wedge d\bar{z}^{\bar{j}}.$$

$0 < \beta \le 1$ ω_0 is C^∞ Kähler metric.

$\mathcal{H}_\beta(\omega) :=$ the space of all Kähler cone potentials φ s.t. $\omega_\varphi := \omega_0 + i\partial\bar{\partial}\varphi$ is a Kähler cone metric.

Hölder space. $C^{k,\alpha,\beta}$.

Hölder exponent α satisfies $\alpha\beta < 1-\beta$.

$k = 0, 1, 2$ Donaldson.

U_p is a coordinate chart

which intersects with the divisor D.

- A function $u(z) : U_p \to \mathbb{R}$

is $C^{0,\alpha,\beta}$, if $v(|z^1|^{\beta-1} z^1, z^2, \cdots, z^n)$

$:= u(z^1, \cdots, z^n)$.

and v is $C^{0,\alpha}$ Hölder function.

- $C^{2,\alpha,\beta} := \{ u \mid u, \partial u, \sqrt{-1}\partial\bar{\partial} u \in C^{0,\alpha,\beta} \}$

$\boxed{\text{[} \nabla^2 u \text{ to be } C^{0,\alpha,\beta} \times \text{]}}$

$k = 3, 4, \cdots$

$C_w^{3,\alpha,\beta}$ is defined to be the space of function φ, which satifies the following conditions.

1) $\varphi \in C^{2,\alpha,\beta}$.

2) $g := g_\varphi$ is the Kähler metric defined by the potential φ.

$$(g_\varphi)_{k\bar{l}} = (g_0)_{k\bar{l}} + (\varphi)_{k\bar{l}}.$$

$$\frac{\partial g_{k\bar{l}}}{\partial z^i}, \quad |z^1|^{1-\beta} \frac{\partial g_{k\bar{l}}}{\partial z^i}, \quad |z^1|^{1-\beta} \frac{\partial g_{1\bar{l}}}{\partial z^i},$$

$$|z^1|^{2-2\beta} \frac{\partial g_{1\bar{1}}}{\partial z^i} \in C^{0,\alpha,\beta}.$$

3) optimal growth rate

$$\boxed{\kappa = \beta - \alpha\beta}$$

$$|z'|^{1-\beta}\frac{\partial g_{k\bar{l}}}{\partial z'}, \quad |z'|^{2-2\beta}\frac{\partial g_{k\bar{1}}}{\partial z'},$$

$$|z'|^{2-2\beta}\frac{\partial g_{1\bar{l}}}{\partial z'}, \quad |z'|^{3-3\beta}\frac{\partial g_{1\bar{1}}}{\partial z'}.$$

are $O(|z'|^{-\kappa})$.

Cone gedesic.

$$\begin{cases} \Omega_{\Phi}^{n+1} = \tau\,\Omega_b^{n+1} & \text{on } \mathcal{M} \\ \Phi = \Phi_0 & \text{on } \partial\mathcal{X} \end{cases}$$

$\mathcal{X} := X \times [0,1] \times S^1$

$\mathcal{M} := M \times [0,1] \times S^1$.

τ is small const.

Thm (cone geodesic).

Suppose ω_1, ω_2 are two Kähler cone metrics in $\mathcal{H}_\beta([\omega_0])$

Then there exists a unique generalised cone geodesic connecting them. Furthermore if φ_1, φ_2 are $\boxed{C_w^{3,\alpha,\beta}}$ Kähler cone potentials then the generalise cone geodesic is $C_w^{1,1,\beta}$.

$$\|\varphi\|_{C_w^{1,1,\beta}} = \sup\Big\{ |\varphi| + |\partial_t \varphi| + |\partial_z \varphi|_\omega + |\partial_z \partial_{\bar{z}} \varphi|_\omega + \left|\frac{\partial^2 \varphi}{\partial z^i \partial t}\right|_\Omega + |s|^k \left|\frac{\partial^2 \varphi}{\partial z^i \partial t}\right|_\Omega + |s|^{2k} \left|\frac{\partial^2 \varphi}{\partial t^2}\right| \Big\}$$

Csck cone metric

Step 1. Reference metric $\boxed{\omega_\theta}$

$$c_1(X,D) := c_1(X) - (1-\beta)c_1(L_D)$$

$$\mathrm{Ric}(\omega_\theta) = \theta + 2\pi(1-\beta)[D].$$ tr.

θ is a smooth $(1,1)$ form $\in c_1(X,D)$.

After a few computation,

$$(*_r)\quad \frac{\omega_\theta^n}{\omega_0^n} = \frac{e^{h_0}}{|S|_h^{2-2\beta}}, \qquad \int_M \omega_\theta^n = V.$$

- $\mathrm{Ric}(\omega_0) = \theta + (1-\beta)\Theta_D + i\partial\bar\partial h_0$.
- $\Theta_D := -i\partial\bar\partial \log h$, h Hermitian metric on L_D.

Lemma. There exists a unique solution $\varphi_\theta \in C^{2,\alpha,\beta}$ for $(*_r)$.

Step 2.

Defn: A csck cone metric $\omega_{csck} := \omega_0 + i\partial\bar\partial \varphi_{csck}$ is a Kähler cone metric with $\boxed{\varphi_{csck} \in C^{2,\alpha,\beta}}$ ① and satisfying the PDE:

$$\begin{cases} \dfrac{\omega_{csck}^n}{\omega_\theta^n} = e^F \quad ② \\ \Delta_{csck} F = \operatorname{tr}_{\omega_{csck}} \theta - \underline{S}_\beta. \end{cases}$$

The const $\underline{S}_\beta$ is independent of the choice of φ_{csck}.

$$\underline{S}_\beta = \frac{C_1(X, D)\,[\omega_0]^{n-1}}{[\omega_0]^n}.$$

Step 3. Regularity.

Defn. $D_\omega^{4,\alpha,\beta}(\omega_\theta) = \{\varphi \in C^{2,\alpha,\beta}, \log\frac{\omega_\varphi^n}{\omega_\theta^n} \in C^{2,\alpha,\beta}\}$.

$C_\omega^{4,\alpha,\beta}(\omega) = \{u \in C^{2,\alpha,\beta}, \Delta_\omega u \in C^{2,\alpha,\beta}\}$.

Thm (Regularity of Csck cone metric)

Suppose that $\omega_\varphi := \omega_0 + i\partial\bar\partial\varphi$ is cscK cone. Then

$$\boxed{\varphi \in C_\omega^{3,\alpha,\beta} \cap D_\omega^{4,\alpha,\beta}(\omega_\theta)}$$

Moreover. φ is geometrically polyhomogeneous.

Energy Functionals.

Log Entropy: $E_\beta(\varphi) := \frac{1}{V}\int_M \log \frac{\omega_\varphi^n}{\omega_\theta^n}\, \omega_\varphi^n$.

D-functional: $D_{\omega_0}(\varphi) := \frac{1}{V}\frac{1}{n+1}\sum_{j=0}^{n}\int_M \varphi\, \omega_0^j \wedge \omega_\varphi^{n-j}$.

$$D(\omega_0, \omega_\varphi) = D(\omega_0, \omega_\psi) + D(\omega_\psi, \omega_\varphi).$$

$$\dot{J}_\chi(\varphi) := \frac{1}{V}\int_M \varphi \sum_{j=0}^{n-1} \omega_0^j \wedge \omega_\varphi^{n-1-j} \wedge \chi.$$

χ is a closed $(1,1)$ form.

$$\underline{\chi} := \frac{n\int_X \chi \wedge \omega_0^{n-1}}{V}$$

$$\boxed{J_\chi(\varphi) := \dot{J}_\chi(\varphi) - \underline{\chi}\cdot \dot{D}_{\omega_0}(\varphi)}.$$

Exe①

$$\partial_t J = \frac{1}{V}\int_M \partial_t \varphi \left(n\chi \wedge \omega_\varphi^{n-1} - \underline{\chi}\, \omega_\varphi^n\right).$$

Exe② $\dot{J}_{i\partial\bar\partial f}(\varphi) = \frac{1}{V}\int_M f\left(\omega_\varphi^n - \omega_0^n\right).$

Log Mabuchi K energy.

$$V_\beta(\varphi) = E_\beta(\varphi) + J_{-\theta}(\varphi) + \frac{1}{V}\int_M (\eta + h_0)\,\omega_0^n$$

$$\eta := -(1-\beta)\log|S|_h^2$$

Prop. The log K energy is continuous generalised and convex along the cone geodesic.

Defn. We set λ to be smooth $\boxed{\text{non-negative}}$ closed $(1,1)$ form.

Defn. λ- twisted cscK cone metric.

$$\begin{cases} F = \log \omega_\varphi^n / \omega_\theta^n \\ \Delta_\varphi F = \mathrm{tr}_\varphi(\theta - \lambda) - (\underline{S}_\beta - \underline{\lambda}) \end{cases}.$$

$$\Rightarrow \quad S_\varphi = \mathrm{tr}_\varphi \lambda + \underline{S}_\beta - \underline{\lambda}.$$

log λ-twisted K energy.

$$\nu_{\beta,\lambda}(\varphi) := \nu_\beta(\varphi) + J_\lambda(\varphi).$$

Prop. The J_λ-functional is strictly convex along $\boxed{C_w^{1,1,\beta}}$-cone geodesic.

$\Rightarrow$ Cor. The λ-twisted K energy is strictly convex along $\boxed{C_w^{1,1,\beta}}$ cone geodesic.

$\Rightarrow$ Cor. The λ-twisted cscK cone metric is unique, and is global mininiser of $\nu_{\beta,\lambda}$.

Approximation of twisted cscK

Step 1 Approximation of ω_θ

$$\frac{\omega_{\theta_\varepsilon}^n}{\omega_0^n} = \frac{e^{h_0+c}}{(|s|_h^2+\varepsilon)^{1-\beta}}; \quad \int_M \omega_{\theta_\varepsilon}^n = V.$$

Exe e^c has explicit formula.

Lemma: $\boxed{Ric(\omega_{\theta_\varepsilon})} \geq \tilde{\theta} := \theta + \min\{(1-\beta)\Theta_D, 0\}$

Step 2.

$$\underline{V^{\varepsilon}_{\beta,\chi}(\varphi)} := \boxed{\frac{1}{V}\int_M \log\frac{\omega_\varphi^n}{\omega_{\theta_\varepsilon}^n}\,\omega_\varphi^n}$$

$$+ J_{-\theta}(\varphi) + J_\chi(\varphi) + \boxed{\frac{1}{V}\int_M [-(1-\beta)\log(|s|_h^2+\varepsilon)]\,\omega_0^n}$$

$$+ \frac{1}{V}\int_M h_0\,\omega_0^n + c.$$

Lemma: $\underline{V^{\varepsilon}_{\beta,\chi}(\varphi) \geq V_{\beta,\chi}(\varphi) - C}.$

Step 3. The critical ponit of $\mathcal{V}^{\varepsilon}_{\beta,\chi}$

is
$$\begin{cases} F_\varepsilon = \log \omega^n_{\varphi_\varepsilon} / \omega^n_{\theta_\varepsilon} & (*)_\varepsilon \\ \Delta_{\varphi_\varepsilon} F_\varepsilon = tr_{\varphi_\varepsilon}(\theta - \chi) - (\underline{S}_\beta - \underline{\chi}) \end{cases}$$

$$S(\omega_{\varphi_\varepsilon}) = tr_{\varphi_\varepsilon}(Ric(\omega_{\theta_\varepsilon}) - \theta + \chi) + \underline{S}_\beta - \underline{\chi}.$$

Prop (Existence of φ_ε for $(*_\varepsilon)$

Assume that χ is C^∞, non-negative, closed $(1,1)$.

$\boxed{C_1(L_D) \geq 0}$.

Assume that $\mathcal{V}_{\beta,\chi}$ is proper.

$\Rightarrow$ $(*)_\varepsilon$ has a unique solution

for any $\varepsilon \in (0, 1]$.

proof: $\mathcal{V}_{\beta,\chi}$ is proper

inequality $\Longrightarrow$ $\boxed{\mathcal{V}^{\varepsilon}_{\beta,\chi} \geq \mathcal{V}_{\beta,\chi} - C}$

$\mathcal{V}^{\varepsilon}_{\beta,\chi}$ is also proper.

$c_1(L_D) \geq 0 \Rightarrow Ric(\omega_{\theta_\varepsilon}) \geq \theta$.

$\Rightarrow Ric(\omega_{\theta_\varepsilon}) - \theta + \chi \geq 0$.

Apply the result of Chen-Cheng to conclud the existence of φ_ε for $(*_\varepsilon)$.

$$
\begin{tikzcd}
\mathcal{V}^{\varepsilon}_{\beta,\chi} \arrow[rr, "\checkmark"] \arrow[d, "\checkmark"] & & \mathcal{V}_{\beta} \arrow[d, "\checkmark"] \\
\varphi^{\varepsilon} \arrow[rr, Rightarrow, "A\ priori\ estimates\ ?", "convergence"'] & & \varphi\ ?
\end{tikzcd}
$$

twisted cscK $\qquad\qquad$ cscK cone

Lemma. (Properness $\Rightarrow$ entropy bound)

$$\sup_{\varepsilon\in(0,1]} E_\beta^\varepsilon(\varphi_\varepsilon) \le C.$$

$$E_\beta^\varepsilon := E_\beta - c - \frac{1-\beta}{V}\int_M \left[\log(|s|_h^2+\varepsilon) - \log|s|_h^2\right]\omega_\varphi^n.$$

proof. φ_ε is twisted cscK.

$\Rightarrow$ φ_ε is Global Minimiser of $\nu_{\beta,\chi}^\varepsilon$.

$$\Rightarrow \nu_{\beta,\chi}^\varepsilon(\varphi_\varepsilon) \le \nu_{\beta,\varepsilon}^\varepsilon(0) = 0$$

$$\Rightarrow \nu_{\beta,\chi}(\varphi_\varepsilon) \le \nu_{\beta,\chi}^\varepsilon(\varphi_\varepsilon) + C.$$

$$\xRightarrow{\text{properness}} d_1(\varphi_\varepsilon, 0) \le \nu_{\beta,\chi}(\varphi_\varepsilon) \le C.$$

$$\xRightarrow{d_1 \sim J} J_{-\theta}(\varphi_\varepsilon),\ J_\chi(\varphi_\varepsilon) \ge C.$$

$$\xRightarrow{\nu_{\beta,\chi}^\varepsilon} E_\beta^\varepsilon(\varphi_\varepsilon) \le C,\ \forall\varepsilon.$$

Lemma (A priori estimates)

For any $\varepsilon \in (0,1]$ there is a C s.t.

$$\|\varphi_\varepsilon\|_\infty, \|F_\varepsilon\|_\infty, \|\partial F_\varepsilon\|_{\omega_{\theta_\varepsilon}'} \leq C$$

$$C^{-1}\omega_{\theta_\varepsilon} \leq \omega_{\varphi_\varepsilon} \leq C\,\omega_{\theta_\varepsilon}.$$

The const C depends on

$\boxed{E_\beta^\varepsilon(\varphi_\varepsilon),}$ $\|\theta - \eta\|_\infty$, $\inf_X \theta$,

α_1, α_β, ε_β, n.

Lemma (Entropy approximation).

Take $\varepsilon \to 0$, the entropy converges.

If $E_\beta(\varphi) < \infty$, $E_\beta^\varepsilon(\varphi_\varepsilon) \to E_\beta(\varphi)$,

and $\sup_{\varepsilon \leq \varepsilon_0} E_\beta^\varepsilon(\varphi_\varepsilon) \leq E_\beta(\varphi) + 1$

when ε_0 is sufficiently small.

proof: $\boxed{\text{Estimates.}}$

Theorem (Properness Theorem).

Assume that the twisted term χ is C^∞, non-negative, closed $(1,1)$. $c_1(L_D) \geq 0$.

Suppose that $\nu_{\beta,\chi}$ is proper.

$\Rightarrow$ there exises a χ-twisted cscK cone metric $\omega_\varphi := \omega_0 + i\partial\bar\partial\varphi$ with $\varphi \in D_w^{4,\alpha,\beta}(\omega_0)$.

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proof: φ_ε is a sequence of smooth approximat solutions $(*)_\varepsilon$.

Take $\varepsilon \to 0$, φ_ε converges smoothly on the regular part $X \setminus D$.

The limit φ satisfies the PDE for twisted cscK cone metri on $X \setminus D$.

A priori estimates

$\Rightarrow \|\varphi\|_\infty, \quad C^{-1}\omega_\theta \le \omega_\varphi \le C\omega_\theta.$

$$\Delta_\varphi F = tr_\varphi(\theta - \chi) - (\underline{S}_\beta - \underline{\lambda}) = RHS$$

$$\in L^\infty.$$

D-N-M $\Rightarrow$ $F \in C^{0,\alpha,\beta}$.

$$\omega_\varphi^n / \omega_\theta^n = e^F$$

E-K $\Rightarrow$ $\varphi \in C^{2,\alpha,\beta}$.

$\Rightarrow$ $RHS \in C^{0,\alpha,\beta}$

by Schauder $\Rightarrow$ $F \in C^{2,\alpha,\beta}$.

$\Rightarrow$ $\varphi \in D_w^{4,\alpha,\beta}(\omega_\theta)$..

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