Lecture No 22 May 24, 2022 (Tue)

24.2 Stochastic heat equation with periodic boundary condition

- We are considering stochastic heat equation with external field $F = \dot{W}(t,x)$, which is a space-time Gaussian white noise:
$$\frac{\partial u}{\partial t} = \Delta u + \dot{W}(t,x), \quad x \in D\,(\subset \mathbb{R}^d).$$
- As we pointed, $\dot{W}(t,x) \in H_{\mathrm{loc}}^{-\frac{d+1}{2}-}(\mathbb{R}_+ \times \mathbb{R}^d)$ a.s. ω.
- We want to give a meaning to this equation and study the regularity of the solution $u(t,x)$.
- We consider under boundary condition.
- First, let us consider on a d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (i.e. on $[0,1]^d$ with periodic boundary condition):
$$\frac{\partial u}{\partial t} = \Delta u + \dot{W}(t,x), \quad x \in \mathbb{T}^d. \tag{1}$$

- Let's consider Fourier series expansion of the equation (1).
- The eigenfunctions of Laplacian Δ on $\mathbb{T}^d$ are products of

$$\{\sqrt{2}\sin \pi n_i x_i, 1, \sqrt{2}\cos \pi m_i x_i\}_{n_i, m_i = 1,2,\dots}$$

  choosing one for each $i = 1, 2, \dots, d$.
- We denote them $\psi_{\mathbf{n}}$ by parametrizing by $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, that is, choose $\sqrt{2}\sin \pi n_i x_i$, 1 and $\sqrt{2}\cos \pi(-n_i)x_i$ for i such that $n_i > 0$, $n_i = 0$ and $n_i < 0$, respectively, and make product to define $\psi_{\mathbf{n}}(x), x \in \mathbb{T}^d$.
- Then, $\psi_{\mathbf{n}}$ is an eigenfunction of Δ with eigenvalue $-\lambda_{\mathbf{n}}$, i.e. $\Delta\psi_{\mathbf{n}} = -\lambda_{\mathbf{n}}\psi_{\mathbf{n}}$, where $\lambda_{\mathbf{n}} = \pi^2|\mathbf{n}|^2 \equiv \pi^2 \sum_{i=1}^d n_i^2$.
- Moreover, $\{\psi_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ is a CONS of $L^2(\mathbb{T}^d)$.
- In particular, one can express $W(t,x)$ with independent 1-dimensional Brownian motions $\{B_t^{\mathbf{n}}\}$ (at least formally) as we discussed before:

$$W(t,x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} B_t^{\mathbf{n}}\psi_{\mathbf{n}}(x).$$

- Noting that $u(t,x)$ might be a generalized function, set

$$u_t^{\mathbf{n}} = \langle u(t), \psi_{\mathbf{n}} \rangle \ (\equiv_{\mathcal{D}'} \langle u(t), \psi_{\mathbf{n}} \rangle_{\mathcal{D}},\ \mathcal{D} = C^\infty(\mathbb{T}^d))$$

$$\left(= (u(t), \psi_{\mathbf{n}})_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} u(t,x)\psi_{\mathbf{n}}(x)dx \ \text{ if } u(t) \in L^2 \right). \tag{2}$$

- Then, expanding (1) based on $\{\psi_{\mathbf{n}}\}$, one would obtain

$$\dot{u}_t^{\mathbf{n}} = -\lambda_{\mathbf{n}} u_t^{\mathbf{n}} + \dot{B}_t^{\mathbf{n}}$$

and this could be understood as 1-dimensional SDE for each $u_t^{\mathbf{n}}$:

$$du_t^{\mathbf{n}} = -\lambda_{\mathbf{n}} u_t^{\mathbf{n}} dt + dB_t^{\mathbf{n}} \tag{3}$$

- This SDE can be easily solved (Duhamel's formula) and

$$u_t^{\mathbf{n}} = e^{-\lambda_{\mathbf{n}} t} u_0^{\mathbf{n}} + \int_0^t e^{-\lambda_{\mathbf{n}}(t-s)} dB_s^{\mathbf{n}} \tag{4}$$

- $u_t^{\mathbf{n}}$ are independent Ornstein-Uhlenbeck processes.

- By (2), we may define the solution $u(t,x)$ of (1) as

$$u(t,x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} u_t^{\mathbf{n}} \psi_{\mathbf{n}}(x) \ \ (\in \mathcal{D}'(\mathbb{T}^d)).$$

- For $s \in \mathbb{R}$, define Sobolev norms of (generalized) functions $u = u(x) \left(= \sum_{\mathbf{n}\in\mathbb{Z}^d} \langle u, \psi_{\mathbf{n}}\rangle \psi_{\mathbf{n}}(x)\right)$ on $\mathbb{T}^d$:

$$\begin{aligned} \|u\|_{H^s} &= \|(-\Delta + 1)^{s/2} u\|_{L^2} \\ &= \left\{ \sum_{\mathbf{n}\in\mathbb{Z}^d} (\pi^2 |\mathbf{n}|^2 + 1)^s \langle u, \psi_{\mathbf{n}}\rangle^2 \right\}^{\frac{1}{2}} \end{aligned}$$

  and Sobolev space $H^s \equiv H^s(\mathbb{T}^d) = \{u; \|u\|_{H^s} < \infty\}$.
- Then, by definition,

$$E[\|u(t)\|_{H^s}^2] = \sum_{\mathbf{n}\in\mathbb{Z}^d} (\pi^2 |\mathbf{n}|^2 + 1)^s E[|u_t^{\mathbf{n}}|^2].$$

- Assuming $u(0) = 0$ (so that $u_0^{\mathbf{n}} = 0$) for simplicity, for $\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, from (4), we have

$$
\begin{aligned}
E[|u_t^{\mathbf{n}}|^2] &= E\left[\left|\int_0^t e^{-\lambda_{\mathbf{n}}(t-s)} dB_s^{\mathbf{n}}\right|^2\right] \\
&\underset{\text{Itô isometry}}{=} \int_0^t e^{-2\lambda_{\mathbf{n}}(t-s)} ds = \frac{1}{2\lambda_{\mathbf{n}}}\left(1 - e^{-2\lambda_{\mathbf{n}} t}\right). \qquad (5)
\end{aligned}
$$

- For $\mathbf{n} = \mathbf{0}$, we have $E[|u_t^{\mathbf{0}}|^2] = E[|B_t^{\mathbf{0}}|^2] = t$.
- Thus, estimating the RHS of (5) from below and above by using $1 - e^{-2t} \le 1 - e^{-2\lambda_{\mathbf{n}} t} \le 1$, we obtain the $L^2(\Omega)$-estimate of the Sobolev norm of the solution $u(t)$ of the stochastic heat equation (1)

$$
\begin{aligned}
&(1 - e^{-2t}) \sum_{\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \frac{(\pi^2 |\mathbf{n}|^2 + 1)^s}{2\lambda_{\mathbf{n}}} + t \\
&\qquad \le E[\|u(t)\|_{H^s}^2] \le \sum_{\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \frac{(\pi^2 |\mathbf{n}|^2 + 1)^s}{2\lambda_{\mathbf{n}}} + t
\end{aligned}
$$

- However, the series in both sides behaves as

$$\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{(\pi^2|\mathbf{n}|^2+1)^s}{2\lambda_{\mathbf{n}}} = \frac{1}{2}\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{(\pi^2|\mathbf{n}|^2+1)^s}{\pi^2|\mathbf{n}|^2}$$

$$\sim \int_{\{x\in\mathbb{R}^d:|x|\geq 1\}} |x|^{2s-2}dx = c_d \int_1^\infty r^{2s-2}r^{d-1}dr$$

- The RHS converges if and only if $s < \frac{2-d}{2}$, i.e.

$$u(t) \in H^{\frac{2-d}{2}-}(\mathbb{T}^d) \text{ a.s.}$$

- This gives the (spatial) regularity of the solution of (1).

- In particular, if $d = 1$, the solution $u(t)$ belongs to the space $H^{\frac{1}{2}-}(\mathbb{T})$, that is, a usual function space.
- On the other hand, if $d \geq 2$, since $\frac{2-d}{2} \leq 0$, $u(t)$ is genuinely a generalized function.
- This is not a problem for linear equation, but for nonlinear equation, one faces a difficulty to give a mathematical meaning if we have nonlinear term, which is definable for usual functions.
- Or the equation becomes ill-posed.

Higher order SPDE

- To compare, let us consider the SPDE with higher order differential operator $-(-\Delta)^m, m \in \mathbb{N}$ instead of Δ:

$$\frac{\partial u}{\partial t} = -(-\Delta)^m u + \dot{W}(t,x), \quad x \in \mathbb{T}^d \tag{6}$$

- $(-\Delta)^m u$ is defined via Fourier series expansion:

$$(-\Delta)^m u = \sum_{\mathbf{n} \in \mathbb{Z}^d} \lambda_{\mathbf{n}}^m \langle u, \psi_{\mathbf{n}} \rangle \psi_{\mathbf{n}}.$$

- Then, assuming $u(0) = 0$, similarly as above for the solution $u(t)$ of (6), we have

$$E[\|u(t)\|_{H^s}^2] \sim \sum_{\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \frac{|\mathbf{n}|^{2s}}{\lambda_{\mathbf{n}}^m} \sim \sum_{\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} |\mathbf{n}|^{2s-2m}$$
$$\sim \int_{\{x \in \mathbb{R}^d : |x| \geq 1\}} |x|^{2s-2m} dx = c_d \int_1^\infty r^{2s-2m} r^{d-1} dr.$$

- The RHS converges if and only if $s < \frac{2m-d}{2}$, i.e. $u(t) \in H^{\frac{2m-d}{2}-}(\mathbb{T}^d)$ a.s.

- In particular, if $d \leq 2m - 1$ ($d < 2m$), the solution stays in a usual function space.
- In other words, as m becomes large, due to the smoothing effect of the operator $-(-\Delta)^m$, the regularity of the solution $u(t)$ is improved.

24.3 Stochastic heat equation with Dirichlet or Neumann boundary conditions

- We are considering stochastic heat equation

$$\frac{\partial u}{\partial t} = \Delta u + \dot{W}(t,x), \quad x \in D\,(\subset \mathbb{R}^d) \tag{7}$$

- As we saw on $\mathbb{T}^d$, the SPDE (7) has a usual function-valued solution only in 1-dimensional space so that let us consider (7) on $I = [0,1]$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \dot{W}(t,x), \quad x \in I \tag{8}$$

- We impose the Dirichlet boundary condition

$$u(t,0) = u(t,1) = 0 \tag{9}$$

or the Neumann boundary condition

$$u'(t,0) = u'(t,1) = 0. \tag{10}$$

- Since the solution is not differentiable even in 1-dimensional space (as we saw $u(t) \in H^{\frac{2-d}{2}-}(\mathbb{T}^d)$ a.s. on $\mathbb{T}^d$), Neumann condition is not definable in classical sense.

Solution in a sense of generalized functions (G-solution in short)

- We introduce two classes of test functions:

$$\Phi_D = \{\varphi \in C^\infty(I); \varphi(0) = \varphi(1) = 0\}$$
$$\Phi_N = \{\varphi \in C^\infty(I); \varphi'(0) = \varphi'(1) = 0\}$$

- For $u \in C^2(I), \varphi \in C^\infty(I)$, integration by parts shows

$$\langle u'', \varphi \rangle = \langle u, \varphi'' \rangle - u\varphi'|_0^1 + u'\varphi|_0^1. \qquad (11)$$

- The boundary terms in the RHS vanishes if $u(0) = u(1) = 0$ and $\varphi \in \Phi_D$ or if $u'(0) = u'(1) = 0$ and $\varphi \in \Phi_N$, where $\langle u, \varphi \rangle = \int_I u(x)\varphi(x)dx$.
- Let $u \in C([0, T], C(I))$ (a.s.). Motivated by the above observation, we call u a solution in generalized functions' sense of (8) with Dirichlet condition if u satisfies (9) and

$$\langle u(t), \varphi \rangle = \langle u(0), \varphi \rangle + \int_0^t \langle u(s), \varphi'' \rangle ds + \int_0^t \int_I \varphi(x) W(dsdx) \quad (12)$$

for $^\forall\varphi \in \Phi_D$, and that of (8) with Neumann condition if u satisfies (12) for $^\forall\varphi \in \Phi_N$.
- Note that (12) is stated in integrated form in t and space.

- We interpret the last term in (12) as

$$\int_0^t \int_I \varphi(x) W(dsdx) := W_t(\varphi)$$

  with the white noise process $W(t)$ or as a stochastic integral.
- We define the equation (8) in integrated form (12) by a similar reason for SDEs.
- For each case, (12) can be extended as

$$\begin{aligned}\langle u(t), \varphi(t) \rangle =& \langle u(0), \varphi(0) \rangle + \int_0^t \langle u(s), \varphi'' + \partial_s \varphi \rangle ds \\ &+ \int_0^t \int_I \varphi(s,x) W(dsdx) \end{aligned} \tag{13}$$

  for ${}^\forall t$-dependent test function $\varphi(t,x) \in C^{1,\infty}([0,T] \times I)$ such that $\varphi(t,\cdot) \in \Phi_D$ for the case of Dirichlet condition and $\varphi(t,\cdot) \in \Phi_N$ for Neumann condition for ${}^\forall t \in [0,T]$.

Mild solutions

- Let $p_D(t,x,y), p_N(t,x,y), t \geq 0, x, y \in I$ be the heat kernel on I with Dirichlet condition and Neumann condition.
- We call

$$u(t,x) = \int_I u(0,y) p_D(t,x,y) dy + \int_0^t \int_I p_D(t-s,x,y) W(dsdy) \tag{14}$$

a mild solution of (8)+(9), and

$$u(t,x) = \int_I u(0,y) p_N(t,x,y) dy + \int_0^t \int_I p_N(t-s,x,y) W(dsdy) \tag{15}$$

a mild solution of (8)+(10).

- The last terms in (14) and (15) are defined as stochastic integrals with respect to the white noise process.
- (14), (15) are obtained by applying Duhamel's formula formally regarding the term $\frac{\partial^2 u}{\partial x^2}$ as a leading term and $\dot{W}(t,x)$ as its perturbation.

- Note that, also under periodic boundary condition, one can rewrite the solution $u(t)$ into a similar form of mild solution with the heat kernel $p_{\mathbb{T}^d}$ on $\mathbb{T}^d$ based on the discussion in §24.2.
- In this sense, mild solutions are naturally introduced.

Equivalence between G-solutions and mild solutions

- Mild solution is G-solution: First, for Dirichlet condition, set $u_1(t,x)$, $u_2(t,x)$ the 1st and 2nd terms in the RHS of (14).
- Then, in terms of stochastic differentials, we have

$$d\langle u(t), \varphi\rangle = \langle \partial_t u_1(t), \varphi\rangle dt + d\langle u_2(t), \varphi\rangle \qquad (16)$$

- Here, $u_1(t,x)$ satisfies the heat equation without noise and Dirichlet condition so that, by noting (11), we have for ${}^\forall \varphi \in \Phi_D$,

$$\langle \partial_t u_1(t), \varphi\rangle = \langle \partial_x^2 u_1(t), \varphi\rangle = \langle u_1(t), \varphi''\rangle \qquad (17)$$

- On the other hand, for $u_2(t)$, in the sense of stochastic differentials, regarding $p_D(0,x,y) = \delta_0(x-y)$, we have

$$
\begin{aligned}
d\langle u_2(t), \varphi\rangle &= d\left\{\int_0^t \int_I \left(\int_I p_D(t-s,x,y)\varphi(x)dx\right) W(dsdy)\right\} \\
&= \int_I \varphi(y)W(dtdy) + \left\{\int_0^t \int_I \left(\int_I \partial_t p_D(t-s,x,y)\varphi(x)dx\right) W(dsdy)\right\}dt \\
&= \int_I \varphi(y)W(dtdy) + \left\{\int_0^t \int_I \left(\int_I p_D(t-s,x,y)\varphi''(x)dx\right) W(dsdy)\right\}dt \\
&= \int_I \varphi(x)W(dtdx) + \langle u_2(t), \varphi''\rangle dt \qquad (18)
\end{aligned}
$$

- We have used Itô's formula for the 2nd equality. For the 3rd equality, we use (11) noting that $\partial_t p_D = \partial_x^2 p_D$ (and symmetry of p_D in x, y), p_D satisfies Dirichlet boundary condition (in x) and $\varphi \in \Phi_D$.

- By inserting (17), (18) into (16), we obtain

$$d\langle u(t), \varphi\rangle = \int_I \varphi(x) W(dtdx) + \langle u(t), \varphi''\rangle dt.$$

- (12) (i.e. G-solution) is its integrated form. Thus, it is shown that the mild solution is a G-solution in Dirichlet case.
- For Neumann condition, taking $\varphi \in \Phi_N$, we can make similar computation.

- Conversely, to show that G-solution is a mild solution in case of the Dirichlet condition, for $^\forall f \in C_0^\infty(I^\circ)$, $I^\circ = (0,1)$, determine $\varphi(t, \cdot) \in \Phi_D$ by

$$\varphi'' + \partial_t \varphi = 0, \quad t \in [0, T]$$
$$\varphi(T) = f.$$

- Indeed, φ is written as $\varphi(t,x) = \int_I f(y) p_D(T-t, x, y) dy$. Insert this into (13) taking $t = T$. Then, we obtain (14) (with $t = T$).
- Similar for Neumann condition. □

§25 Examples of SPDEs

Stochastic reaction-diffusion equation:

- We take $F = f(u) + \dot{W}(t,x)$ in $(\star)$ and consider (formally)

$$\frac{\partial u}{\partial t} = \Delta u + f(u) + \dot{W}(t,x), \quad x \in \mathbb{R}^d \text{ or } D.$$

- This type of SPDE appears in several different context, called
  - Stochastic reaction-diffusion equation,
  - Stochastic Allen-Cahn equation,
  - Time-dependent Ginzburg-Landau equation,
  - Dynamic $P(\phi)$-model (stochastic quantization, Parisi-Wu)
- We will discuss this equation in the rest of the course.

$$\frac{\partial u}{\partial t} = \Delta u + F, \quad x \in \mathbb{R}^d \qquad (\star)$$

Stochastic Navier-Stokes equation:

- Equation for the turbulence:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p + \dot{W}(t,x) = F, \ \ x \in \mathbb{R}^d \\ \operatorname{div} u = 0. \end{cases}$$

- $\nu > 0$ is the viscosity, $u = u(t,x,\omega) \in \mathbb{R}^d$ is the velocity field of fluid, $p = p(t,x,\omega) \in \mathbb{R}$ is the pressure, $F = F(t,x,\omega) \in \mathbb{R}^d$ is the external force.
- $\dot{W}(t,x)$ represents a fluctuation of the external force, which is usually taken as colored noise, but when $d = 2$, Da Prato-Debussche studied the case of the space-time Gaussian white noise.

KPZ (Kardar-Parisi-Zhang) equation:

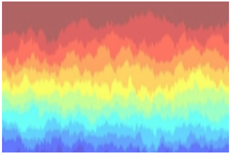
- KPZ equation describes growing interfaces with fluctuation. For the height of interfaces $h(t,x)$,

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h + \tfrac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \quad x \in \mathbb{R}$$

  Or renormalized KPZ equation might have the meaning:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left\{\left(\partial_x h\right)^2 - \delta_x(x)\right\} + \dot{W}(t,x), \quad x \in \mathbb{R}.$$

- Linear case (without $\frac{1}{2}(\partial_x h)^2$): $h(t,x) \in C^{\frac{1}{4}-,\frac{1}{2}-}$ a.s.
- We discussed this equation in the mini-course of Nov-Dec, 2020 at Yau Center.



Random motion of strings or curves:

- Equation for random curve in $\mathbb{R}^d$: $\sigma \in [0,1] \mapsto f(\sigma) \in \mathbb{R}^d$

$$\frac{\partial}{\partial t} f_t(\sigma) = \frac{\partial^2}{\partial \sigma^2} f_t(\sigma) + b(f_t(\sigma)) + \alpha(f_t(\sigma))\dot{W}(t,\sigma), \quad \sigma \in [0,1]$$

  where $\alpha : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \to \mathbb{R}^d$,
  $\dot{W}(t,\sigma) = \{\dot{W}^\alpha(t,\sigma)\}_{\alpha=1}^d$ is $\mathbb{R}^d$-valued space-time Gaussian white noise on $[0,1]$.
- This can be extended to the manifold M-valued equation:

$$\frac{\partial}{\partial t} f_t(\sigma) = \boldsymbol{\Delta} f_t(\sigma) + V_0(f_t(\sigma)) + \circ\dot{W}(t,\sigma,f_t(\sigma)), \quad \sigma \in \mathbb{T}^1,$$

  where $\boldsymbol{\Delta} f(\sigma) = \{(\boldsymbol{\Delta} f)^\alpha(\sigma)\}_{\alpha=1}^d \in T_{f(\sigma)}M$,

$$(\boldsymbol{\Delta} f)^\alpha = \frac{\partial^2 f^\alpha}{\partial \sigma^2} + \Gamma^\alpha_{\beta\gamma}(f(\sigma))\frac{\partial f^\beta}{\partial \sigma}\frac{\partial f^\gamma}{\partial \sigma}$$

  and $\{\Gamma^\alpha_{\beta\gamma}\}$ is a Christoffel symbol determined by the Riemannian metric on M, V_0 is a vector field and noise is defined in Stratonovich's sense; see recent results by Hairer and others.

Parabolic Anderson model:

- Heat equation with random potential $V(x, \omega)$

$$\frac{\partial u}{\partial t} = \Delta u + V(x, \omega) u$$

SPDE in population genetics:

- Equation for density of population

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + \sqrt{u(t, x)}\dot{W}(t, x), \ x \in \mathbb{R},$$
$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + \sqrt{u(t, x)(1 - u(t, x))}\dot{W}(t, x), \ x \in \mathbb{R}$$

- These SPDEs have multiplicative noises.