PART I. $SL_2(\mathbb{F}_q)$

Chapter 1. Structure

Chapter 2. Harish-Chandra induction

TODAY (Chapter 3. Introduction to ℓ-adic cohomology

Chapter 4. Deligne-Lusztig theory for $SL_2(\mathbb{F}_q)$

Chapter 3. "Introduction" to ℓ-adic cohomology

3.A. $\mathbb{F}_q$-structure on varieties.

- An $\mathbb{F}_q$-structure on an $\mathbb{F}$-vector space M is an $\mathbb{F}_q$-vector subspace M_q such that $M = \mathbb{F} \otimes_{\mathbb{F}_q} M_q$

  → always exists

  → unique up to auto.

- An $\mathbb{F}_q$-structure on an $\mathbb{F}$-algebra A is an $\mathbb{F}_q$-subalgebra A_q such that $A = \mathbb{F} \otimes_{\mathbb{F}_q} A_q$

  → does not exist in general

  → is not unique in general

Example 3.1. If $A = \mathbb{F}[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r\rangle$ with $f_i \in \mathbb{F}_q[X_1, \ldots, X_n]$, then take $A_q = \mathbb{F}_q[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r\rangle$. ■

Example 3.2. Take $A = \mathbb{F}[X,Y]/\langle Y^2 - X^3 - 1\rangle$ $(p \neq 2, 3)$. Set $Y' = Y/\sqrt{3}$.

$$A = \mathbb{F}[X, Y']/\langle 3Y'^2 - X^3 - 1\rangle$$

⇝ Two $\mathbb{F}_q$-structures: A_q, A'_q

But $A_7 \not\simeq A'_7$

$$\mathrm{Hom}(A_7, \mathbb{F}_7) \xrightarrow{\sim} \{(x,y) \in \mathbb{F}_7 \times \mathbb{F}_7 \mid y^2 = x^3 + 1\}$$

$\downarrow$

card $= 11$

$$\mathrm{Hom}(A'_7, \mathbb{F}_7) \xrightarrow{\sim} \{(x,y') \in \mathbb{F}_7 \times \mathbb{F}_7 \mid 3y'^2 = x^3 + 1\}$$

$\downarrow$

card $= 3$. ■

Let A be an $\mathbb{F}$-algebra (commutative). Then

$$a \longmapsto a^q$$

is an $\mathbb{F}_q$-linear endomorphism of A (not $\mathbb{F}$-linear, not an automorphism in general).

Definition. An $\mathbb{F}$-linear endomorphism $F^* : A \to A$ is called a (geometric) Frobenius endomorphism over $\mathbb{F}_q$ if $\{a \in A \mid F^*(a) = a^q\}$ is an $\mathbb{F}_q$-structure on A.

Assume that A has an $\mathbb{F}_q$-structure A_q.

Let

$$F^* : A = \mathbb{F} \otimes_{\mathbb{F}_q} A_q \longrightarrow A$$
$$\lambda \otimes a \longmapsto \lambda a^q$$

Then (easy)

(3.3) $\quad A_q = \{a \in A \mid F^*(a) = a^q\}$

and so F^* is a Frob. endo. $/\mathbb{F}_q$.

<u>Varieties</u>. In this course, a variety is a quasi-projective algebraic variety over $\mathbb{F}$.

$$X \xhookrightarrow{\text{open}} \overline{X} \xhookrightarrow{\text{closed}} \mathbb{P}^n(\mathbb{F})$$

An $\mathbb{F}_q$-structure on X is determined by a "Frobenius endomorphism" $F : X \to X$ which is a morphism such that there exists an affine covering $(U_i)_{i \in I}$ of X such that U_i is F-stable and $F^* : \mathcal{O}_X(U_i) \longrightarrow \mathcal{O}_X(U_i)$ is a Frobenius endomorphism $/\mathbb{F}_q$.

<u>Example 3.4.</u> (1) $F : \mathbb{A}^n(\mathbb{F}) \longrightarrow \mathbb{A}^n(\mathbb{F})$, $(x_1, \dots, x_n) \longmapsto (x_1^q, \dots, x_n^q)$ is a Frob. endo. $/\mathbb{F}_q$.

Indeed

$$\{P \in \mathbb{F}[X_1, \dots, X_n] \mid P(X_1^q, \dots, X_n^q) = P(X_1, \dots, X_n)^q\}$$
$$= \mathbb{F}_q[X_1, \dots, X_n].$$

(2) By glueing (1), $F : \mathbb{P}^n(\mathbb{F}) \longrightarrow \mathbb{P}^n(\mathbb{F})$, $[x_0, \dots, x_n] \longmapsto [x_0^q, \dots, x_n^q]$ is a Frob. endo. $/\mathbb{F}_q$.

Theorem 3.5. Let $F : X \to X$ be a Frob. endo. $/\mathbb{F}_q$.

(a) If X' is an F-stable locally closed subvariety of X, then $F|_{X'} : X' \to X'$ is a Frob. endo. $/\mathbb{F}_q$.

(b) If $\varphi \in \mathrm{Aut}(X)$ such that $(\varphi F)^m = F^n$ for some $n \geq 1$, then $\varphi F : X \to X$ is another Frob. endo. $/\mathbb{F}_q$.

(c) If $F' : X \to X$ is another Frob. endo. $/\mathbb{F}_q$, then $F'^n = F^n$ for some $n \geq 1$.

(d) F^n is a Frob. endo. $/\mathbb{F}_{q^n}$.

Proof. See Digne - Michel (second edition, Prop. 4.1.11). ∎

$$X^F = \{x \in X \mid F(x) = x\}$$
$$\underset{\text{def}}{=} X(\mathbb{F}_q)$$

ambiguous notation

Points with coordinates in $\mathbb{F}_q$

3.B. Counting points.

Artin conjecture (1924), Hasse theorem (1933?)

Let $\mathcal{E}$ be a smooth elliptic curve defined over $\mathbb{F}_q$ (e.g.
$$\mathcal{E} = \{[x:y:z] \in \mathbb{P}^2(\mathbb{F}) \mid y^2 z = x^3 + axz^2 + bz^3\}$$
with $a, b \in \mathbb{F}_q$ and $4a^2 + 27b^3 \neq 0$).
Then
$$\big|\underbrace{\#}_{\text{cardinality}} \mathcal{E}(\mathbb{F}_q) - 1 - q\big| \leq 2\sqrt{q}$$

Weil theorem (1949). Let $\mathcal{C}$ be a smooth irreducible projective curve over $\mathbb{F}_q$ of genus g, then
$$|\# \mathcal{C}(\mathbb{F}_q) - 1 - q| \leq 2g\sqrt{q}$$

Hasse's proof uses the group structure.

Weil's proof uses Riemann-Roch theorem (Serre duality).

"defined over $\mathbb{F}_q$"
$\Updownarrow$
" admits an $\mathbb{F}_q$-structure"

Ramanujan τ-function:

$$\sum_{n\geq 1} \tau(n) z^n = z \prod_{n\geq 1} (1-z^n)^{24}$$

Ramanujan conjecture (1916)
Deligne theorem (1974)

$$p \text{ prime} \Rightarrow |\tau(p)| \leq 2\, p^{11/2}$$

(Serre: $\cancel{\forall p,\ |\tau(p)| \leq c\, p^{9/2}}$)

Funny fact:

$$\tau(n) \equiv \sigma_{11}(n) \mod 691$$

$$\text{where } \sigma_k(n) = \sum_{d|n} d^k$$

Note that $\zeta(12) = \dfrac{691\ \pi^{12}}{638\ 512\ 875}$

Example. $X = \mathbb{P}^d(\mathbb{F})$;

$$\# X(\mathbb{F}_{q^n}) = 1 + q^n + \dots + q^{(d-1)n} + q^{dn}$$

$$Z(X,t) = \exp\left(\sum_{n\geq 1} \sum_{j=0}^{d} \frac{q^{nj}}{n} t^n \right)$$

$$= \exp\left(-\sum_{j=0}^{d} \operatorname{Log}(1-q^j t)\right) = \frac{1}{(1-t)(1-qt)\cdots(1-q^d t)}$$

3.C. Weil conjectures

For X an algebraic variety defined over $\mathbb{F}_q$, set

$$Z(X,t) = \exp\left(\sum_{n\geq 1} \frac{\# X(\mathbb{F}_{q^n})}{n} t^n \right)$$

Weil conjectures (1949), Grothendieck ('60s) - Deligne (1974) Theorem.

Assume that X is smooth, irreducible and projective. Then:

Dwork 1959 (a) Rationality: $Z(X,t) = \mathbb{Q}(t)$ (for any X)

$d = \dim X$

(b) Functional equation:

$$Z\left(X, \frac{1}{q^d t}\right) = \pm q^{\frac{dE}{2}} t^E Z(X,t)$$

$$\left(E = \sum (-1)^i \beta_i\right)$$

(c) Riemann hypothesis:

$$Z(X,t) = \frac{P_1(t)\, P_3(t) \cdots P_{2d-1}(t)}{P_0(t)\, P_2(t) \cdots P_{2d}(t)}$$

with $P_0(t) = 1-t$; $P_{2d}(t) = 1 - q^d t$

and $P_i(t) = \prod_{j=1}^{\beta_i} (1 - \alpha_{ij} t)$ with $|\alpha_{ij}| = q^{i/2}$.

(d) If X is obtained by "reasonable" reduction modulo p of some complex variety $\mathcal{X}$, then

$$\beta_i = \dim H^i(\mathcal{X})$$

3.D. ℓ-adic cohomology.

Let X be a variety over $\mathbb{F}$.

Assume that we are given:

- An action of a finite group Γ
- A Frob. endo. $F : X \to X$ over $\mathbb{F}_q$.

Grothendieck (60's). $\ell \neq p$ prime

$\leadsto H^i_c(X)$: $\overline{\mathbb{Q}}_\ell$-vector space

+ action of Γ

+ action of F

Does there exist a cohomology theory over $\mathbb{Q}$ such that $H^i_c(X) = \overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}} H^i_c(X, \mathbb{Q})$

?

__Theorem 3.7__ (Grothendieck). Assume that $\dim X = d$ and let $\mathcal{J}(X)$ denote the set of its irreducible components of dim. d.

(a) $\dim H^i_c(X) < \infty$

(b) $H^i_c(X) = 0$ if $i < 0$ or $i > 2d$

(c) $H^{2d}_c(X) \underset{\overline{\mathbb{Q}}_\ell \Gamma\text{-mod}}{\simeq} \overline{\mathbb{Q}}_\ell[\mathcal{J}(X)]$ (permutation module)

(d) If $\mathcal{U} \subset X$ is open, Γ-stable, F-stable and $\mathcal{Z} = X \setminus \mathcal{U}$, we have a long exact sequence

$$\cdots \to H^i_c(\mathcal{U}) \to H^i_c(X) \to H^i_c(\mathcal{Z}) \to H^{i+1}_c(\mathcal{U}) \to \cdots$$

(e) Künneth formula:

$$H^n_c(X \times X') = \bigoplus_{i=0}^{n} H^i_c(X) \otimes H^{n-i}_c(X')$$

(f) Poincaré duality: if X is smooth, irreducible, projective, then there is a Γ-equivariant, F-equivariant duality

$$H^i_c(X) \times H^{2d-i}_c(X) \longrightarrow H^{2d}_c(X) \simeq \overline{\mathbb{Q}}_\ell$$

(g) $H^i_c(\mathbb{A}^d(\mathbb{F})) = \begin{cases} \overline{\mathbb{Q}}_\ell & \text{if } i = 2d \\ 0 & \text{if } i \neq 2d. \end{cases}$

(h) If X is smooth, irreducible, affine, then $H^i_c(X) = 0$ if $i < d$.

(i) $H^i_c(X)^\Gamma = H^i_c(X/\Gamma)$

(j) If Γ is contained in a connected algebraic group acting regularly on X, then Γ acts trivially on $H^i_c(X)$

__Example 3.8__ $X = \mathbb{P}^1(\mathbb{F}) = \underset{\mathcal{U}}{\underset{\|}{\mathbb{A}^1(\mathbb{F})}} \cup \underset{\mathcal{Z}}{\underset{\|}{\{\infty\}}}$

$$0 \to \overset{=0}{H^0_c(\mathcal{U})} \to H^0_c(X) \to H^0_c(\mathcal{Z}) \simeq \overline{\mathbb{Q}}_\ell$$

$$\to \overset{=0}{H^1_c(\mathcal{U})} \to H^1_c(X) \to H^1_c(\mathcal{Z}) = 0$$

$$\to \underset{\overline{\mathbb{Q}}_\ell}{\underset{\|}{H^2_c(\mathcal{U})}} \to \underset{\overline{\mathbb{Q}}_\ell}{\underset{\|}{H^2_c(X)}} \to \underset{=0}{H^2_c(\mathcal{Z})} \to 0$$

$\dim H^0_c(\mathbb{P}^1(\mathbb{F})) = 1$

$\dim H^1_c(\mathbb{P}^1(\mathbb{F})) = 0$

$\dim H^2_c(\mathbb{P}^1(\mathbb{F})) = 1$. ∎

__Exercise.__ Cohomology of $\mathbb{P}^d(\mathbb{F})$?

Cohomology of $\mathbb{A}^1(\mathbb{F}) \setminus \{0\}$

Example 3.9.

$\mathbb{K} = \overline{\mathbb{Q}}_\ell$

$$X = \mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q) = \mathbb{A}^1(\mathbb{F}) \setminus \mathbb{A}^1(\mathbb{F}_q)$$

smooth affine

$G = SL_2(q)$ acts on X.

$\mathbb{K}_G \oplus St = \overline{\mathbb{Q}}_\ell[\mathbb{P}^1(\mathbb{F}_q)]$

Action of G: Example 2.7

By 3.7(d), we have:

$$0 \to \underbrace{\cancel{H^0_c(X)}}_{\text{by (R)}\ 0} \longrightarrow \underbrace{H^0_c(\mathbb{P}^1(\mathbb{F}))}_{= \mathbb{K}_G\ (\simeq \overline{\mathbb{Q}}_\ell)} \longrightarrow H^0_c(\mathbb{P}^1(\mathbb{F}_q)) = \mathbb{K}_G \oplus St \quad (c)$$

$$\to H^1_c(X) \longrightarrow \underbrace{\cancel{H^1_c(\mathbb{P}^1(\mathbb{F}))}}_{0} \longrightarrow H^1_c(\mathbb{P}^1(\mathbb{F}_q)) = 0$$

$$\to H^2_c(X) \longrightarrow \underbrace{H^2_c(\mathbb{P}^1(\mathbb{F}))}_{= \mathbb{K}_G\ (= \overline{\mathbb{Q}}_\ell)} \longrightarrow \underbrace{\cancel{H^2_c(\mathbb{P}^1(\mathbb{F}_q))}}_{0} \to 0$$

How G acts on $H^i_c(\mathbb{P}^1(\mathbb{F}))$? $G \subset SL_2(\mathbb{F})$ acts on $\mathbb{P}^1(\mathbb{F})$

ANSWER: TRIVIALLY !!

$$0 \longrightarrow \mathbb{K}_G \longrightarrow \mathbb{K}_G \oplus St \longrightarrow H^1_c(X) \longrightarrow 0$$

$$\longrightarrow 0 \longrightarrow H^2_c(X) \longrightarrow \mathbb{K}_G \longrightarrow 0$$

As representations of G:

$$H^1_c(X) \simeq St$$

$$H^2_c(X) \simeq \mathbb{K}_G.$$