2020-10-13 Kähler geometry

<u>Def</u> (M, L) polarized mfd

$\underset{\text{def}}{\Longleftrightarrow}$ A pair of complex mfd M and an <u>ample</u> line bundle on M.

$\underline{c_1(L)} > 0$.

($\Rightarrow$ M algebraic mfd by Kodaira embedding)

$$\boxed{\text{Stability can be interpreted in terms of symplectic geometry using "moment map". (Kempf-Ness).}}$$

Let (N, Ω) be a symplectic mfd i.e. Ω satisfies the following

- Ω closed 2-form.
- $\Omega : TN \times TN \longrightarrow \mathbb{R}$ non-degenerate

i.e. if $\Omega(X, Y) = 0$ for ${}^\forall Y$ then $X = 0$.

Ω is called a symplectic form.

<u>Example</u> If (M, ω) is Kähler, ω Kähler form. $(M, \omega) = (N, \Omega)$

Let K be a Lie group acting (N, Ω) as symplectomorphisms
(symplectic diffeomorphism)
i.e. $\forall g \in K$, $g^* \Omega = \Omega$.

Let k be the Lie algebra of K. Then $\forall X \in k$ defines a vector field on N which we denote by the same letter X

and $L_X \Omega = 0$. L_X Lie derivative

$$\frac{d}{dt}\Big|_{t=0} \underbrace{\mathrm{Exp}(tX)^* \Omega}_{\Omega} = L_X \Omega$$

$$\quad = 0$$

Since $L_X = d \circ i(X) + i(X) \circ d$, $d\Omega = 0$.
we have $0 = L_X \Omega = d(i(X)\Omega)$.

So $i(X)\Omega$ is closed.

<u>Def</u> $X \in \mathfrak{k}$ is a Hamiltonian vector field

if $i(X)\Omega$ is <u>exact</u>, i.e.

$\exists\, u_X \in C^\infty(N)$ s.t. $i(X)\Omega = -du_X$.

u_X is called the Hamiltonian function.

(Instead you can start with $u \in C^\infty(N)$
then $\exists\, X$ s.t. $i(X)\Omega = -du$
$X = X_u$ is called the Hamiltonian
v.f. of u.)

<u>Def</u> Let $\mathfrak{k}^*$ be the dual of $\mathfrak{k}$.

$\mu : N \longrightarrow \mathfrak{k}^*$ is a <u>moment map</u> for the action of K

$\underset{\text{def}}{\iff}$ (1) $\langle d\mu, X \rangle = -i(X)\Omega$

(2) $g^*\mu = \mu \circ Ad(g)$: K-equivariant.

$\mu_{g(p)}(X) = \mu_p\left(Ad(g)X\right)$

Set $u_X(p) = \langle \mu(p), X \rangle$. Then

$$du_X = \langle d\mu, X \rangle = -i(X)\Omega$$

$\therefore$ u_X is a Hamiltonian function.

Suppose $[\Omega] \in H^2_{DR}(N)$ is an integral class.

$$\left(\begin{array}{l} 0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0 \\ H^2(N, \mathbb{Z}) \to H^2(N ; \mathbb{R}) \cong H^2_{DR}(N) \\ [\Omega] \in \text{Image of this map} \end{array} \right)$$

$\exists$ $L \to N$ complex line bundle with

$$\underbrace{c_1(L)} = [\Omega].$$

Let ∇ be a connection of L.

e local frame.

$\nabla e = e \cdot \theta$ $\qquad$ θ local 1-form.
(connection form)

Suppose $\frac{1}{2\pi} d\theta = \Omega$.

(General fact: If $[\Omega]$ is an integral class then such L and ∇ always exist.)

Let $L^* = L -$ zero section.

locally on U $\quad L^*|_U \cong U \times \mathbb{C}^*$

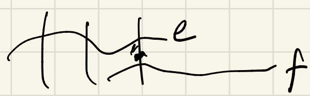
$$z\,e(p) \longleftrightarrow (p, z)$$

$z \neq 0$

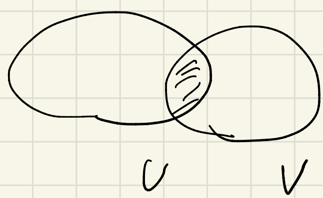
$$\tilde{\theta} := \frac{dz}{z} + \theta$$

Homework $\tilde{\theta}$ defines a global 1-form on L^*.

(This is the connection form for the principal $\mathbb{C}^*$-bundle L^*. $\frac{dz}{z}$ is the Maurer-Cartan form.)



$f = e\sigma \qquad \sigma \in C^\infty(U \cap V, \mathbb{C}^*)$



$= ez \qquad = fw.$

$p = z e = wf = e\sigma w \rightsquigarrow z = \sigma w$

$\nabla e = e\theta$ on U, $\qquad \nabla f = f\theta'$ on V.

$$\nabla f = \nabla e \cdot \sigma + e \cdot d\sigma = e\theta\sigma + e\,d\sigma$$

$$\nabla f = e\sigma \cdot \theta'$$

$$\theta\sigma + d\sigma = \sigma\theta'$$

$$\theta' = \theta + \sigma^{-1} d\sigma.$$

$$\frac{dw}{w} + \theta' = \frac{d(\sigma^{-1} z)}{\sigma^{-1} z} + \theta + \sigma^{-1} d\sigma$$

$$= \frac{dz}{z} - \cancel{\frac{\sigma^{-1} d\sigma\, \sigma^{-1}}{\sigma^{-1}}} + \theta + \cancel{\sigma^{-1} d\sigma}$$

$$= \frac{dz}{z} + \theta.$$

Homework.

It is generally true that for e.g. vector bundle, connection gives a global 1-form on its associated principal bundle. P_{GL}

P_{GL} = the set of all frames of E_p for all $p \in N$.

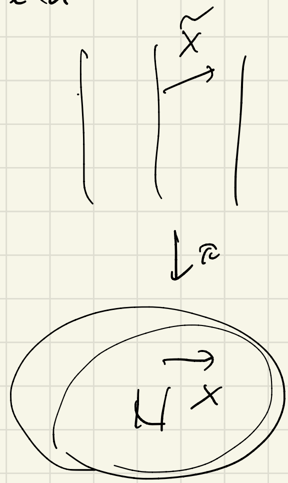
= "frame bundle"

"Kobayashi–Nomizu: Theory of connections"

Then $$d\tilde{\theta} = d(d\log z + \theta) = \overset{\pi^*}{d\theta} = \overset{\pi^*}{2\pi\Omega}$$

$$\mathbb{C}^* \xrightarrow{\pi} N$$

Suppose the action of K lifts to L.
Then $X \in k$ defines a vector field $\tilde{X}$ on L^*.



$$i(\tilde{X})\, d\tilde{\theta} = i(\tilde{X})\, \pi^* d\theta$$
$$= i(\pi_* \tilde{X})\, d\theta$$
$$= i(X)\, d\theta = 2\pi\Omega.$$

Suppose also $\tilde{\theta}$ is K-invariant.

(This can be done if K is compact.)

$$L_X \tilde{\theta} = 0.$$

$$\therefore\ d\, i(\tilde{X})\tilde{\theta} + i(\tilde{X})\, d\tilde{\theta} = 0.$$

$$\therefore\ d(\tilde{\theta}(\tilde{X})) = -\, i(X)\Omega$$

Conclusion $\tilde{\theta}(\tilde{X})$ is a Hamiltonian function for $X \in k$ descends to M. (Homework.).

Prop. $\mu : \mathcal{H} \to \mathfrak{k}^*$, $\quad p \mapsto \mu(p)$

$GL(n,\mathbb{R})$

$GL(n,\mathbb{C})$

$$\langle \mu(p), X \rangle = \hat{\theta}(\tilde{X})(p)$$

is a moment map.

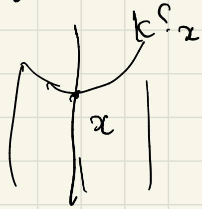
(Homework : Prove the equivariance.)

Theorem (Kempf-Ness). $\Leftarrow$ finite dim set-up.

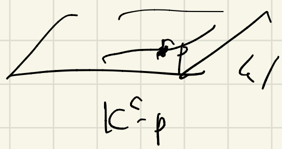
Let K^c be the complexification of K.

$p \in N$ is polystable w.r.t. K^c-action

$\Longleftrightarrow$ $\mu^{-1}(0) \cap K^c \cdot p \neq \phi$.



" K^c-orbit of p contains a zero of the moment map"



cscK

$\mathcal{H}$ = the space of Kähler forms $\check{J}_J$

($\doteqdot$ space of almost complex str.)

$$\underset{(J)}{\mu(\omega)} = \underline{\text{scalar curvature}} \in \mathfrak{k}^*$$

$$\mathfrak{k} = \mathfrak{g} = \left\{ u_x \;\middle|\; \begin{matrix} x \text{ hol} \\ \int u_x \omega^n = 0 \end{matrix} \right\}$$

$$\mu(\omega) = 0 \iff \mathrm{Scal}(\omega) = \mathrm{const.}$$

$$\int \underline{\mathrm{Scal}}\, u_X = \mathrm{Scal} \int u_X = 0.$$

$$\mathcal{K}: \underset{\omega}{p} = \text{space of Kähler forms in } [\omega].$$

Proof of Thm.

General fact (1). Hilbert-Mumford criterion.

Stabi