The LYZ equation

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1. Definition.

- At the most fundamental level, mirror symmetry describes a framework for relating complex geometry to symplectic geometry on two Calabi-Yau manifolds.

- One particular aspect of mirror symmetry is the relationship between the derived category of coherent sheaves on one manifold, and Fukaya's category of Lagrangian submanifolds with local systems on the other.

- In the simple case of a torus, C. Leung, S.-T. Yau and E. Zaslow gave an explicit formulation of an equation for a connection A on a holomorphic line bundle that corresponds to the special Lagrangian (SL) equation on the mirror.

- The equation on connection A is

$$\mathrm{Im}(\omega - F)^n = \tan\hat{\theta}\ \mathrm{Re}(\omega - F)^n, \tag{1}$$

where ω is the Kähler form, F is the curvature of the connection A, and $\hat{\theta}$ is the phase of the sL.

- This equation is called the deformed Hermitian Yang-Mills (d-HYM) equation in the literature. We call it the Leung-Yau-Zaslow (LYZ) equation instead of the dHYM equation.

- Such an equation is also derived from supersymmetry by M. Marino, R. Minasian, G. Moore, and A. Strominger.

- Now let (X,ω) be a compact Kähler manifold of complex dimension n and χ a closed real $(1,1)$-form on X.

- The LYZ equation on (X,ω,χ) is

$$\mathrm{Re}(\chi_u + \sqrt{-1}\omega)^n = \cot\hat{\theta}\ \mathrm{Im}(\chi_u + \sqrt{-1}\omega)^n. \tag{2}$$

- Here

$$\chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u$$

for a real smooth function u on M and $\hat{\theta}$ is the argument of the complex number

$$\int_X (\chi + \sqrt{-1}\omega)^n.$$

- Let

$$\lambda = (\lambda_1, \ldots, \lambda_n)$$

be the eigenvalues of χ_u with respect to ω. Let

$$\theta_i = \mathrm{arccot}\lambda_i \quad \text{for } 1 \le i \le n.$$

- Then $\lambda_i = \cot\theta_i$ and we can compute as follows:

$$\begin{aligned}
&(\chi_u + \sqrt{-1}\omega)^n \\
=&\prod_{i=1}^{n} (\lambda_i + \sqrt{-1})\,\omega^n \\
=&\frac{\exp\left(\sqrt{-1}\sum_{i=1}^{n}\theta_i\right)}{\prod_{i=1}^{n}\sin\theta_i}\,\omega^n \\
=&\frac{\cos\left(\sum_{i=1}^{n}\theta_i\right)}{\prod_{i=1}^{n}\sin\theta_i}\omega^n + \sqrt{-1}\frac{\sin\left(\sum_{i=1}^{n}\theta_i\right)}{\prod_{i=1}^{n}\sin\theta_i}\omega^n.
\end{aligned}$$

- So the LYZ equation becomes

$$\cos\Big(\sum_{i=1}^{n}\theta_i\Big) = \cot\hat{\theta}\sin\Big(\sum_{i=1}^{n}\theta_i\Big),$$

or

$$\cot\Big(\sum_{i=1}^{n}\theta_i\Big) = \cot\hat{\theta}.$$

- If we define

$$\theta_\omega(\chi_u) := \sum_{i=1}^{n}\theta_i = \sum_{i=1}^{n}\operatorname{arccot}\lambda_i,$$

then the LYZ equation is simply written as

$$\theta_\omega(\chi_u) = \theta_0. \tag{3}$$

- It is called supercritical if $\theta_0 \in (0,\pi)$ and hypercritical if $\theta_0 \in (0,\frac{\pi}{2})$.

- In this formulation it is clear that the equation is the complex version of the special Lagrangian equation for graphs, with θ_0 the analog of the Lagrangian angle and λ_j the analogs of the eigenvalues of the Hessian of the generating function of graph.

- The LYZ equation has been extensively studied by many mathematicians.

- In 2014, Jacob and Yau initiated to study the LYZ equation.

A. Jacob, S.-T. Yau. A special Lagrangian type equation for holomorphic line bundles. Math. Ann. 369 (2017), 869-898. arXiv:1411.7457.

2. Elliptic cases.

- When $n = 2$, Jacob and Yau proved that the solution exists if and only if there exists a function $\underline{u}$ such that

$$\chi_{\underline{u}} > \cot\theta_0\,\omega.$$

- Denote

$$\alpha = \chi_{\underline{u}} - \cot\theta_0\,\omega.$$

Then $\alpha > 0$ and it defines a Kähler metric.

- Jacob and Yau then translated the LYZ equation into the complex Monge-Ampère equation

$$(\alpha + i\partial\bar{\partial}u)^2 = \frac{1}{\sin^2\theta_0}\omega^2$$

which was solved by Yau in 1976.

- When $n \geq 3$, Collins, Jacob and Yau solved the LYZ equation for $\theta_0 \in (0, \pi)$ by assuming that the following two conditions hold.

(1) There exists a subsolution $\underline{u}$, which means that $\chi_{\underline{u}}$ satisfies the inequality

$$\max_{1 \leq j \leq n} \sum_{i \neq j} \operatorname{arccot} \lambda_i(\chi_{\underline{u}}) < \theta_0 (< \pi). \tag{4}$$

(2) The function $\underline{u}$ also satisfies the inequality:

$$\theta_\omega(\chi_{\underline{u}}) \Big(= \sum_{i=1}^{n} \operatorname{arccot} \lambda_i(\chi_{\underline{u}}) \Big) < \pi. \tag{5}$$

T. Collins, A. Jacob, S.-T. Yau. $(1,1)$ forms with specified Lagrangian phase: a priori estimates and algebraic obstructions. Camb. J. Math. 8 (2020), 407-452. arXiv:1508.01934.

- To be precise, Collins-Jacob-Yau proved the following

- **Theorem 1.** (Collins-Jacob-Yau) Let (M, ω) be a compact Kähler manifold of dimension n and χ a closed real $(1,1)$-form on M with $\theta_0 \in (0, \pi)$. Suppose there exists a subsolution $\underline{u}$ of LYZ equation (2) in the sense of (4) and $\underline{u}$ also satisfies inequality (5). Then there exists a unique smooth solution of the LYZ equation.

- The definition of subsolutions of fully nonlinear equations was introduced by B. Guan.

B. Guan. Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. Duke Math. J. 163(2014), 1491-1524.

- G. Székelyhidi gave an equivalent version and Collins-Jacob-Yau used it to the LYZ equation which is equivalent to inequality (4).

G. Székelyhidi. Fully non-linear elliptic equations on compact Hermitian manifolds. J. Differential Geom. 109(2018), 337-378.

- L. Huang, J. Zhang and X. Zhang also considered the solution on a compact almost Hermitian manifold for the case $\theta_0 \in (0, \frac{\pi}{2})$.

  L. Huang, J. Zhang, X. Zhang. The deformed Hermitian-Yang-Mills equation on almost Hermitian manifolds. Sci. China Math. 65(2021), 127-152. arXiv:2011.14091.

- C.-M. Lin generalized Collins-Jacob-Yau's result to the Hermitian manifold (M, ω) with $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$.

  C.-M. Lin. Deformed Hermitian-Yang-Mills equation on compact Hermitian manifolds. arXiv:2012.00487.

- The natural question is whether condition (5) is superfluous.

- When $n = 3$, Pingali can solve the LYZ equation without condition (5) by translating the LYZ equation into a mixed Monge-Ampère type equation.

V. P. Pingali. The deformed Hermitian Yang-Mills equation on three-folds. arXiv:1910.01870.

- When $n = 3$ and $n = 4$, C.-M. Lin can solve the LYZ equation without condition (5).

C.-M. Lin. The deformed Hermitian-Yang-Mills equation, the positivstellensatz, and the solvability. arXiv:2201.01438v2.

- Very recently, C.-M. Lin claimed that he solved the LYZ equation without condition (5).

C.-M. Lin. On the Solvability of General Inverse σ_k Equations. arXiv:2310.05339.

- **Theorem 2.** (Lin) Let (M, ω) be a compact Kähler manifold of dimension n and χ a closed real $(1,1)$-form on M with $\theta_0 \in (0, \pi)$. If there exists a subsolution $\underline{u}$ of LYZ equation (2) in the sense of (4), then there exists a unique smooth solution of the LYZ equation.

3. Algebro-Geometric condition.

- As analogue of the Nakai-Moishezon type criterion for a Kähler class by Demailly-Paun and for the J-equation by Lejmi-Székelyhidi, there is also a Nakai-Moishezon type criterion for the existence of solution to the LYZ equation.

J.-P. Demailly, M. Paun. Numerical characterization of the Kähler cone. Ann. of Math. (2) 159 (2004), 1247-1274.

M. Lejmi, G. Székelyhidi. The J-flow and stability. arXiv:1309.2821v1.

- Especially, Demailly and Paun used Yau's solution to the Calabi conjecture to give a differential geometric proof of the Nakai-Moishezon numerical criterion for ample line bundles in algebraic geometry.

- Given a Kähler metric ω, denote

$$\Omega_\omega(\chi, p) = \mathrm{Re}(\chi + \sqrt{-1}\omega)^p - \cot\theta_0 \mathrm{Im}(\chi + \sqrt{-1}\omega)^p$$

- Define

$$\mathcal{P}_\omega = \{[\chi] \in H^{1,1}_{\mathbb{R}}(M) \mid \int_Y \Omega_\omega(\chi, p) > 0 \text{ for any } p\text{-subvarity } Y,\ 0 < p < n\}$$

$$\mathcal{S}_\omega = \{[\chi] \in H^{1,1}_{\mathbb{R}}(M) \mid \text{the LYZ equation has a smooth solution on } [\chi]\}.$$

- **Conjecture.** (Collins-Jacob-Yau) For any compact Kähler manifold (M, ω),

$$\mathcal{P}_\omega = \mathcal{S}_\omega.$$

- G. Chen made an important progress on the CJY conjecture.

- A smooth family χ_t, $t \in [0, \infty)$ of real closed $(1,1)$-forms is called a test family if and only if all the following conditions hold.

  (A) When $t = 0$, $\chi_0 = \chi$.

  (B) For all $s > t$, $\chi_s - \chi_t$ is positive definite.

  (C) There exists a large enough number $T \geq 0$ such that for all $t \geq T$, $\chi_t - \cot \frac{\theta_0}{n} \omega$ is positive definite.

G. Chen. The J-equation and the supercritical deformed Hermitian-Yang-Mills equation. Invent. Math. **225** (2021), 529-602.

- **Theorem 3.** (Chen) Let (X, ω) be a compact Kähler manifold of dimension n and χ a closed real $(1,1)$-form on X with $\theta_0 \in (0, \pi)$. Then the follwing statements are equivalent.

  (1) There exists a smooth solution of LYZ equation (2).

  (2) For any smooth test family χ_t, there exists a constant $\varepsilon_{1,1} > 0$ such that for any $t \geq 0$ and p-dimensional subvariety Y,

$$\int_Y \Omega_\omega(\chi_t, p) \geq (n-p)\varepsilon_{1,1} \int_Y \omega^p.$$

  (3) There exist a test family χ_t and a constant $\varepsilon_{1,1} > 0$ such that for any $t \geq 0$ and p-dimensional subvariety Y,

$$\int_Y \Omega_\omega(\chi_t, p) \geq (n-p)\varepsilon_{1,1} \int_Y \omega^p.$$

- G. Chen also made a major progress on the J-equation. Then J. Song proved the Nakai-Moishezon criterion for the J-equation.

J. Song. Nakai-Moishezon criterions for complex Hessian equations. arxiv: 2012.07956.

- Motivated by G. Chen and J. Song, Chu-Lee-Takahashi established the following theorem for the LYZ equation.

- **Theorem 4.** (Chu-Lee-Takahashi) The LYZ equation on a compact Kähler manifold (M, ω) with complex dimension n is solvable for $\theta_0 \in (0, \pi)$ if and only if there exists a Kähler metric γ on M such that for any $1 \leq p \leq n$,

$$\int_M \Omega_\omega(\chi, p) \wedge \gamma^{n-p} \geq 0$$

  and for any proper m-dimensional subvariety Y of M and $1 \leq p \leq m$,

$$\int_Y \Omega_\omega(\chi, p) \wedge \gamma^{m-p} > 0.$$

- Chu-Lee-Takahashi then confirmed the CJY conjecture for projective manifolds.

J. Chu, M.-C. Lee, R. Takahashi. A Nakai-Moishezon type criterion for supercritical deformed Hermitian-Yang-Mills equation. arxiv:2105.10725.

- Junsheng Zhang proved that $\mathcal{S}_\omega$ is a both open and closed subset of $\mathcal{P}_\omega$. He then disproved the CJY conjecture. The counter-example is the blow up of $\mathbb{C}^3/\Lambda$ at a point. Here $\mathbb{C}^3/\Lambda$ is a three dimension torus without any positive dimensional proper analytic subvariety.

J. Zhang. A note on the supercritical deformed Hermitian-Yang-Mills equation. arXiv:2302.06592.

- The situation is very much like the description of the Kähler cone of a compact Kähler manifold by Demailly and Păun.

- Collins and Yau also developed the infinite dimensional GIT picture for the LYZ equation.

T. Collins and S.-T. Yau. Moment maps, nonlinear PDE, and stability in mirror symmetry. arXiv:1811.04824.

- They described an infinite dimensional symplectic manifold, admitting an action by a group of symplectomorphisms, together with a space $\mathcal{H} \subset C^\infty(X, \mathbb{R})$ and a Riemannian structure on $\mathcal{H}$, which can be thought of as analogous to G/K in the finite dimensional GIT.

- In the hypercritical phase case they proved the existence of smooth approximate geodesics, and weak geodesics with $C^{1,\alpha}$ regularity. This is accomplished by proving sharp with respect to scale estimates for the Lagrangian phase operator on collapsing manifolds with boundary.

- They applied these results to the infinite dimensional GIT problem for LYZ.

- They associated algebraic invariants to certain birational models of $X \times \triangle$, where $\triangle \subset \mathbb{C}$ is a disk. Using the existence of regular weak geodesics they proved that these invariants give rise to obstructions to the existence of solutions to the LYZ equation.

4. Parabolic cases.

- For the parabolic flow method, there are also several results.

- Jacob-Yau and Collins-Jacob-Yau proposed the line bundle mean curvature flow

$$\begin{cases} u_t = \theta_0 - \theta_\omega(\chi_u) \\ u(0) = \underline{u} \end{cases} \tag{6}$$

  where $\underline{u}$ is a subsolution of the LYZ equation.

- They proved the existence and convergence of the long-time solution to this flow for the case $\theta_0 \in (0, \frac{\pi}{2})$ and $\theta_\omega(\chi_{\underline{u}}) \in (0, \frac{\pi}{2})$.

- X. Han and X. Jin considered the stability result of this flow.

- Takahashi proposed the tangent Lagrangian phase flow

$$\begin{cases} u_t = \tan(\theta_0 - \theta_\omega(\chi_u)) \\ u(0) = \underline{u} \end{cases} \tag{7}$$

where $\underline{u}$ is a subsolution of the LYZ equation.

R. Takahashi. Tan-concavity property for Lagrangian phase operators and applications to the tangent Lagrangian phase flow. Internat. J. Math. 31 (2020), 26 pp.

- He proved the existence and convergence of the long-time solution to this flow for the case $\theta_0 \in (0, \frac{\pi}{2})$ and $\theta_\omega(\chi_{\underline{u}}) - \theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

- Motivated by the concavity of $\cot\theta_\omega(\chi_u)$ by G. Chen, Yau, Dekai Zhang and myself consider a new flow

$$\begin{cases} u_t = \cot\theta_\omega(\chi_u) - \cot\theta_0, \\ u(x,0) = u_0(x) \end{cases} \tag{8}$$

with

$$\theta_\omega(\chi_{u_0}) < \pi.$$

J. Fu, S.-T. Yau and Dekai Zhang (张德凯). A deformed Hermitian Yang-Mills flow. Accepted for publication in JDG. arXiv:2105.13576v3.

- We first prove the existence theorem of the long-time solution to flow (8).

- **Theorem 5.** (F.-Yau-Zhang) Let (M, ω) be a compact Kähler manifold and χ a closed real $(1,1)$-form with $\theta_0 \in (0, \pi)$. If u_0 satisfies the inequality

$$\theta_\omega(\chi_{u_0}) < \pi,$$

then flow (8) has a unique smooth long-time solution u.

- Then we consider the convergence of the long-time solution to flow (8).

- **Theorem 6.** (F.-Yau-Zhang) Let (M, ω) be a compact Kähler manifold and χ a closed real $(1,1)$ form with $\theta_0 \in (0, \pi)$. Suppose that there exists a subsolution $\underline{u}$ of LYZ equation (3) in the sense of (4) which also satisfies (5). Then there exists a long-time solution $u(x,t)$ of flow (8) with $u_0 = \underline{u}$ and it converges to a smooth solution u^∞ to the LYZ equation:

$$\theta_\omega(\chi_{u^\infty}) = \theta_0.$$

- Hence we reprove the Collins-Jacob-Yau's existence theorem. Our parabolic version seems rather natural and the proof is simpler. In fact, Collins, Jacob and Yau used a double method of continuity.

- The advantage of the new flow is that the imaginary part of the Calabi-Yau functional is constant along the flow, which is the key to do the C^0 estimate.

- We in fact identified which assumption is needed for the long-time existence part and which one is needed for the convergence part.

- The second motivation of our paper is to use flow (8) to study the LYZ equation under the existence of a semi-subsolution:

$$\max_{1\le j\le n}\sum_{i\neq j}\operatorname{arccot}\lambda_i(\chi_{\underline{u}})\le\theta_0(<\pi).$$

- We can solve the two dimensional case. In this case, a smooth function $\underline{u}$ is called a semi-subsolution of the LYZ equation if

$$\chi_{\underline{u}}\ge\cot\theta_0\,\omega. \tag{9}$$

  Hence the LYZ equation may be degenerate.

- We hope our flow is useful to solve the higher dimensional case. This is analog to the semi-stable case of the HYM equation. The HYM flow proposed by Donaldson is very useful in study of the HYM equation in the semi-stable case.

Happy Birthday to Professor Yau!

祝丘先生生日快乐！