Conformal Invariance in 2D Lattice Models

Part 2: Random Cluster Model

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Part 1: Bernoulli Percolation
Part 2: Random Cluster Model
Part 3: Ising Model

Bernoulli percolation vs. FK-percolation

Bernoulli percolation Independent percolation	FK percolation dependent percolation
FKG inequality	True for $q \geq 1$
Phase transition	True for $q \geq 1$
Critical value : $p_c = p_{sd}$	True for $q \geq 1$.
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FK-percolation—definition

Fortuin and Kasteleyn

FK-percolation : also called random-cluster model. It is a generalization of Bernoulli percolation where there is dependence between edges.

- $G = (V, E)$ is a finite graph
- configuration $\omega \in \{0,1\}^E$, $o(\omega)$, $c(\omega)$, $k(\omega)$
- edge-parameter $p \in [0,1]$, cluster-parameter $q > 0$

FK-percolation on G is the probability measure defined by

$$\phi_{p,q,G}[\omega] \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}.$$

FK-percolation—boundary conditions

Fix a partition ξ of ∂G, and identify the vertices in ∂G that belong to the same component of ξ. FK-percolation on G with parameters (p, q) and boundary conditions ξ is the probability measure :

$$\phi^{\xi}_{p,q,G}[\omega] \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}.$$

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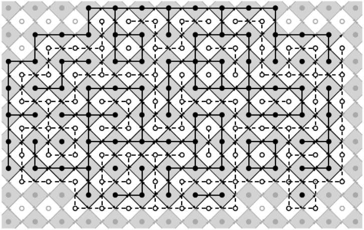
- wired-b.c. : $\phi^{1}_{p,q,G}$
- free-b.c. : $\phi^{0}_{p,q,G}$
- Dobrushin-b.c.
- induced by a config. outside G



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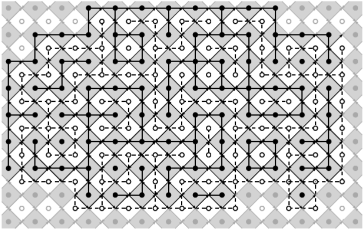

Domain Markov Property

Suppose $G' \subset G$, for any $\psi \in \{0,1\}^{E(G)\setminus E(G')}$,

$$\phi^{\xi}_{p,q,G}[X \mid \omega_e = \psi_e, \forall e \in E(G)\setminus E(G')] = \phi^{\psi^{\xi}}_{p,q,G'}[X].$$

Theorem (FKG Inequality)

Fix $p \in [0,1]$, $q \geq 1$, a finite graph G and some boundary conditions ξ. For any two increasing events A and B, we have

$$\phi^{\xi}_{p,q,G}[A \cap B] \geq \phi^{\xi}_{p,q,G}[A]\phi^{\xi}_{p,q,G}[B].$$

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- Given two proba. measures μ_1, μ_2, we write $\mu_1 \leq_{st} \mu_2$, if $\mu_1[A] \leq \mu_2[A]$ for all increasing event A.
- A proba. measure μ strictly positive if $\mu[\omega] > 0$ for all ω.

Theorem (Holley inequality)

Let μ_1, μ_2 be strictly positive probability measures on the finite state space such that

$$\mu_2[\omega^e]\mu_1[\eta_e] \geq \mu_2[\omega_e]\mu_1[\eta^e], \quad \forall e \in E, \forall \eta \leq \omega.$$

Then $\mu_1 \leq_{st} \mu_2$.

FKG Inequality : consequences

Corollary (Monotonicity)

Fix $p \le p'$ and $q \ge 1$, a finite graph G and some b.c. ξ. We have $\phi^{\xi}_{p,q,G} \le_{st} \phi^{\xi}_{p',q,G}$.

Corollary (Comparison between boundary conditions)

Fix $p \in [0,1]$ and $q \ge 1$, a finite graph G. For any b.c. $\xi \le \psi$, we have $\phi^{\xi}_{p,q,G} \le_{st} \phi^{\psi}_{p,q,G}$.
In particular, for any b.c. ξ, we have $\phi^{0}_{p,q,G} \le_{st} \phi^{\xi}_{p,q,G} \le_{st} \phi^{1}_{p,q,G}$.

Corollary (Finite-energy property)

Fix $p \in [0,1]$ and $q \ge 1$, a finite graph G, and some b.c. ξ, we have

$$\frac{p}{p+(1-p)q} \le \phi^{\xi}_{p,q,G}[\omega(f)=1 \mid \omega(e)=\psi(e)\ \forall e \in E(G)\setminus\{f\}] \le p.$$

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Infinite volume measure

Let ξ_n be a sequence of b.c. The sequence $\phi^{\xi_n}_{p,q,\Lambda_n}$ is said to converge to the infinite-volume measure $\phi_{p,q}$ if

$$\lim_n \phi^{\xi_n}_{p,q,\Lambda_n}[A] = \phi_{p,q}[A],$$

for any event A depending only on the status of finitely many edges.

Proposition

Fix $p \in [0,1]$ and $q \geq 1$. There exist two (possibly equal) infinite-volume random-cluster measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ such that for any event A depending on a finite number of edges,

$$\lim_{n\to\infty} \phi^1_{p,q,\Lambda_n}[A] = \phi^1_{p,q}[A], \quad \lim_{n\to\infty} \phi^0_{p,q,\Lambda_n}[A] = \phi^0_{p,q}[A].$$

Ergodicity

Lemma

Fix $q \geq 1$. The infinite-volume measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are translation invariant and are ergodic.

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The measures ϕ^0 and ϕ^1 are extremal :

$$\phi^0 \leq_{st} \phi \leq_{st} \phi^1.$$

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Question

Do we have $\phi^0 = \phi^1$?

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Phase transition

Theorem

Fix $q \geq 1$. There exists a critical point $p_c = p_c(q) \in [0, 1]$ such that

- *For $p > p_c$, any infinite-volume measure has an infinite cluster almost surely.*
- *For $p < p_c$, any infinite-volume measure has no infinite cluster almost surely.*

Lemma

Fix $q \geq 1$. we have $\phi^0_{p,q} = \phi^1_{p,q}$ for all but countably many values of p.

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Bernoulli percolation	FK percolation
Independent percolation	dependent percolation
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Critical Value : self-dual point

Theorem

Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \geq 1$. The critical value p_c is given by

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Proposition

The dual configuration of the random-cluster model on G with parameters (p, q) and b.c. ξ is the random-cluster model with parameters (p^*, q) on G^* with b.c. ξ^* where $p^* = p^*(p, q)$ satisfying

$$\frac{pp^*}{(1-p)(1-p^*)} = q.$$

Critical Value

Lemma

Fix $q \geq 1$, we have

$$\phi^0_{p_{sd}(q),q}[0 \leftrightarrow \infty] = 0.$$

Theorem

Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \geq 1$.

- *If $p < p_c$, then there exists $c = c(p) > 0$ such that for every $n \geq 1$, $\phi^1_{p,q,\Lambda_n}[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}$.*
- *If $p > p_c$, then there exists $C > 0$ such that $\phi^1_{p,q}[0 \longleftrightarrow \infty] \geq C(p - p_c)$.*

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Continuity of the phase transition

Theorem

- *Fix* $1 \leq q \leq 4$*, we have*

$$\phi^1_{p_c,q}[0 \longleftrightarrow \infty] = 0.$$

- *Fix* $q > 4$*, we have*

$$\phi^1_{p_c,q}[0 \longleftrightarrow \infty] > 0, \quad \phi^0_{p_c,q}[0 \longleftrightarrow \infty] = 0$$

Consequence

- When $1 \leq q \leq 4$, we have $\phi^1 = \phi^0$, and continuous PT.
- When $q > 4$, we have $\phi^1_{p_c,q} \neq \phi^0_{p_c,q}$, and discontinuous PT for $\phi^1_{p_c,q}$.

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