$t-t^*$ equations of Cecotti and Vafa in the context of $N=2$ SQM. It is related to the theory of Primitive form by Kioji Saito and it is a part of mirror symmetry construction.

I will work today purely on QM grounds

$N=2$ SQM based on chiral and antichiral multiplets. Like in $N=1$ SQM one may see that $\underline{Q_+ = \bar\partial + \partial W}$, $Q_- = \partial + \bar\partial \bar W$

δQ_+ correspond to δW, and are Q_--exact.

$\delta Q_+ = [Q_-, \delta W]$ this together with harmonic theory gives an interesting mathematical structure.

Plan of my talk:

I will describe two connections on the space of extended cohomology, and I will compare them.

As a result, I will get interesting quadratic equations.

Let me introduce a spectral parameter z (this par. has different names in the literature, someone call it $\hbar$ - do not like, someone call it $\left(\frac{\partial}{\partial t_0}\right)^{-1}$)

$$Q_z = Q_+ + z Q_-$$

Q_z would act in spaces: $V = \Omega^\cdot_X \otimes \mathbb{C}[z]$

$$\hat V = \Omega^\cdot_X \otimes \mathbb{C}[z, z^{-1}]$$

z is acting just by multiplication in these spaces.

$$W_t = W + \sum_k t_k \underline{\Phi_k}:$$

W is a holom. function on X -

so X is noncompact, φ_K are also hol. functions, t_K are parameters.

<u>Example 1:</u> $X = \mathbb{C}$ noncompact.

$W = x^n$, where x is a hol. coord on X

$\varphi_K = x^K$

This example is called singularity theory

<u>Example 2:</u> $X = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$

$$W = x + \frac{1}{x}$$

called mirror symmetry example

I want $\left|\frac{\partial W}{\partial x}\right|^2 \to \infty$ when x goes along noncompact directions to get discrete spectrum of the $H = -\Delta + \left|\frac{\partial W}{\partial x}\right|^2 + \dots$ to have harmonic theory.

Study H_{Q_z} over the base $\mathbb{C}[[t_1, \dots, t_n]]$

formal disc

H_{Q_z} in the space $\hat{V}$.

It is a bundle and it has two connections

One connection is very easy and it does not involve harmonic theory. For some reason it is called Gauss-Manin connection

$Q_z = Q_+ + z Q_-$, if $Q_+ \to Q_+ + [Q_-, \Phi]$
then it may be compensated by
$\omega \to \omega + \frac{1}{z} \Phi \omega$, where
ω is a representative of Q_z coh.

$$Q_z \omega = 0$$

$$Q_z \to Q_z + [Q_-, \Phi \cdot \epsilon]$$

$$(Q_z + [Q_-, \epsilon\Phi])(\omega + \epsilon\,\delta\omega) = 0$$

$$\underline{\underline{[Q_-, \Phi]\,\omega}} + \underset{\underline{\underline{(Q_+ + z Q_-)\,\delta\omega = 0}}}{\underset{\shortparallel}{Q_z}}\,\delta\omega = 0$$

$$\delta\omega = \frac{1}{z} \Phi \cdot \omega$$

$$e^{+\frac{1}{z}\epsilon\Phi} (Q_+ + [Q_-, \epsilon\Phi] + z Q_-)\, e^{-\frac{1}{z}\epsilon\Phi} = Q_+ + z Q_-$$

This deformation can be rotated away by conjugation with $e^{\frac{1}{z}\epsilon\Phi}$

Here I assumed that $[Q_+, \Phi] = 0$

ω_{GM} – horisontal section of the GM. connection
with the formula on representatives
$\overset{GM}{\delta}\omega = \frac{1}{z} \Phi \omega$: <u>Connection 1.</u>

However, there is also another connection on Q_z cohomology, related to Hodge theory.

Idea of this connection:
while Q_+ is changing, Q_- is not changing, and Harmonic forms of H identify H_{Q_+} with H_{Q_-}
So there is a connection that is constant on H_{Q_-} upon this identification.

How to write def.
$\delta^{Hodge} w$ due to this connection
since class in H_{Q_-} is not changing
$\delta^{Hodge} w$ should be Q_--exact.

$$Q_+(\delta^{Hodge} w) = [Q_+\Phi] \cdot w , \quad \delta^{Hodge} w = Q_-(*)$$

$$\overset{\shortparallel}{Q_-(\Phi w)}$$ since I assume $Q_- w = 0$

Let me try to take h - homotopy to Q_+

$$\{Q_+, h\} = 1 - \mathrm{Proj}_{Harm}$$ ↑ harmonic forms

$$\delta^{Hodge} w_{harm} \overset{?}{=} - Q_- h(\Phi w)$$

$$Q_+ \delta^{Hodge} w = + Q_+ Q_- h(\Phi w) =$$

$$= - Q_- Q_+ h \Phi w = - Q_- (1 - \mathrm{Proj}_{Harm}) \Phi w \pm \underbrace{Q_- h\, Q_+(\Phi w)}_{=0}$$

$$Q_- \mathrm{Proj}_{Harm} \overset{\shortparallel}{0}$$

$$= - Q_- \Phi w$$ ← this is what I wanted.

this is a second integrable connection.
Integrable – no curvature – since it has a global definition given above.

Then I may extend δ^{hodge} to the full space $\hat{V}$ just by tensoring by $\mathbb{C}[z,z^{-1}]$

If we have one integrable connection – – we just trivialize the bundle – – no new local information.
However, if we have two integrable connections we may trivialize bundle by first connection and write the second connection as a 1-form.
(difference of two connections is a 1-form with values in the End of the trivialized vector space)

$$\nabla^{GM} = \nabla^{Hodge} + A, \quad \text{with equations}$$

$$\rightarrow \quad \nabla^{GM} = d + A \rightarrow \quad dA + A^2 = 0$$

So we need to compare them

$$\delta^{\text{hodge}} = Q_- h \Phi w$$

$$\delta^{GM} = \frac{1}{z} \Phi w$$

EZ case suppose, that Φw is zero in Q_+ cohomology then their difference is Q_z-exact

Really, in this case consider

$$Q_z\left(\frac{1}{z} h \varphi w\right) = \frac{1}{z} \underbrace{Q_+ h}_{} \varphi w + \cancel{z} \cdot \frac{1}{\cancel{z}} \underbrace{Q_- h \varphi w}_{\sim}$$

$$\underbrace{\frac{1}{z} \varphi w}_{\sim}$$

However, these two connections are different, if φw is nonzero in Q_+-cohom.

in such case $Q_+ h \varphi w$ is actually

$$\underline{\frac{1}{z} \varphi w - \frac{1}{z} \mathrm{Proj}_{Harm} \varphi w} \longleftrightarrow$$ crutial difference

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Let us write what happens in coordinates

$$W + \varphi_k t_k = W(t) \qquad \frac{\partial W}{\partial t_i} = \varphi_i$$

Let us introduce the following operator:

$\hat{C}_i \to$ action of φ_i on H_{Q_+} cohomology.

Then from our considerations we get

$$\nabla^{GM} = \nabla^H + \frac{1}{z} \hat{C}_i \, dt^i$$

Trivializing Hodge connection GM connection takes the form

$$\nabla^{GM} = d + \frac{1}{z} \hat{C}_i \, dt^i \qquad d \longleftrightarrow dt^i \frac{\partial}{\partial t^i}$$

$$(\nabla^{GM})^2 = 0 \implies$$

(1) $[\hat{C}_i, \hat{C}_j] = 0$ coeff. in front of $\frac{1}{z^2}$

(2) $\partial_j \hat{C}_i - \partial_i \hat{C}_j = 0$ $\leftarrow$ coef in front of $\frac{1}{z}$

(1) and (2) are quite nontrivial equations

we may solve (2) by writing

$\hat{c}_i = \frac{\partial \hat{\tau}}{\partial t_i}$ for some matrix-valued function $\hat{\tau}$

Then (1) looks very nontrivial

(c) $[d\hat{\tau}, d\hat{\tau}] = 0$ (may be called commutativity equations)

Interestingly, solutions to (c) equations may be classified by means of integrable equations theory.

There is an interesting example of solutions to such equations.

<u>Example 1.</u> $W = X^n + \sum_k t_k X^k$

How to do hodge theory?

It turns out that harmonic theory may be replaced by theory of holomorphic germs of harmonic forms at 0:

ω_h is a complicated 1-form containing $\omega_h(x, \bar{x}, dx, d\bar{x})$

In particular, for $n=2$ it is an oscillator and

$$\omega_h = e^{-|x|^2}(dx + d\bar{x})$$

however, holomorphic germ ($\bar{x} \to 0$, keeping x-fixed) is just dx, and the full theory may be played on holomorphic germs.

In particular, $W = x^n$, holomorphic germs of harmonic forms are

$dx, xdx, \ldots, x^{n-2}dx$

hodge connection is

the dif. between GM and hodge

$$\delta^{hodge}\omega^{germ} = d\left(\frac{\varphi.\omega - [\varphi.\omega]^h}{dW}\right)$$

$$\varphi.\omega = \underbrace{\sum_{k=0}^{n-2} x^k dx}_{[\varphi.\omega]^h} + \underbrace{dW\cdot(*)}_{\text{this piece}}$$

It is possible to write the global formula.

$$\omega_{har} = d\left(W^{\frac{k}{n}}\right)_+ , \text{ where } W^{\frac{k}{n}} \text{ - it is a seria } \underbrace{x^k + c_{k-1}x^{k-1} + \ldots + c_0}_{\left(W^{\frac{k}{n}}\right)_+ \text{ only this part}} + c_{-1}x^{-1} + \ldots$$

harmonic germs look like.

Related integrable system is known as a

dispersionless limit of n-reduced KP equations.