Perturbation problems for extremal Kähler metrics

Lars Martin Sektnan

University of Gothenburg

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Today's goal: Explain some background on Kähler geometry, focusing on

- constant scalar curvature Kähler (cscK) metrics;
- extremal Kähler metrics;
- some obstructions to the existence of extremal metrics;
- perturbation problems for extremal metrics.

A lot of the material follows the book by Gábor Székelyhidi.

The cscK problem

Hermitian metrics

Let $X = (M, J)$ be a complex manifold. A Riemannian metric g is *Hermitian* if

$$J^* g = g,$$

which means that

$$g(J(\cdot), J(\cdot)) = g(\cdot, \cdot).$$

I.e., multiplication by i is an isometry on each tangent space.
If g is Hermitian, then

$$\omega(\cdot, \cdot) = g(J(\cdot), \cdot)$$

is a 2-form, as $J^2 = -\mathrm{Id}$:

$$\omega(u, v) = g(Ju, v) = J^* g(Ju, v) = g(J^2 u, Jv) = -g(Jv, u) = -\omega(v, u).$$

The Kähler condition

The metric g is *Kähler* if

$$d\omega = 0.$$

Note that ω and g determine one another. So we may say Kähler metric ω when we really mean Kähler form ω, etc.
The Kähler condition means that X simultaneously has the structures of: a complex manifold, a Riemannian manifold and a symplectic manifold, in a compatible way.

The Kähler condition

There are many other important equivalent conditions to the Kähler condition. For example, g is Kähler if and only if for every point in X, there exists "holomorphic normal coordinates", i.e. coordinates $(z^1, \ldots, z^n)$ such that

$$\omega = i \sum_{j,\bar{k}} g_{j\bar{k}} dz^j \wedge d\bar{z}^k,$$

where

$$g_{j,\bar{k}} = \delta_{j,k} + O(|z|^2).$$

Note that one can always achieve $g_{j,\bar{k}} = \delta_{j,k} + O(|z|)$, as we can choose coordinates to make the metric the Euclidean one at a given point. The Kähler condition is precisely what is needed to be able to choose coordinates in which the $O(|z|)$ terms cancel.

The Kähler condition

By the equation

$$d\omega = 0.$$

if g is Kähler, there is then an associated class

$$\Omega = [\omega] \in H^2(X, \mathbb{R}),$$

the Kähler class of ω. By the $i\partial\bar{\partial}$ Lemma, if $\omega' \in \Omega$ is another Kähler form in the same class, then there exists $\phi : X \to \mathbb{R}$ such that

$$\omega' = \omega + i\partial\bar{\partial}\phi =: \omega_\phi.$$

Canonical metrics

The set

$$\{\phi : \omega_\phi \text{ is a Kähler form}\} \subseteq_{\text{open}} C^\infty(X)$$

is an open subset in the set of smooth functions on X that parametrises Kähler metrics in the class Ω. In particular, in any Kähler class, there is an infinite dimensional set of Kähler metrics. The following question is therefore very natural:

Question: Is there a canonical representative $\omega_\phi \in \Omega$?

Canonical metrics

To begin to answer this, we first need to ask ourselves: what is a good notion of canonical metric?

$\dim_{\mathbb{C}} X = 1$: The uniformisation theorem gives a unique metric of constant curvature. This gives a good canonical choice in complex dimension 1.

$\dim_{\mathbb{C}} X > 1$: Should also be some curvature property, but there are many curvature notions, leading to different notions of canonical metrics!

Ricci curvature

The Ricci curvature Ric_g also satisfies

$$J^* \operatorname{Ric}_g = \operatorname{Ric}_g .$$

So we get a 2-form $\rho_g = \rho_\omega$, the *Ricci form*.
Local expression: If

$$\omega = i \sum_{j,\bar{k}} g_{j\bar{k}} dz^j \wedge d\bar{z}^k,$$

then

$$\begin{aligned} \rho_\omega &= -i\partial\bar{\partial}\big(\log \det(g_{p\bar{q}})\big) \\ &= -i \sum_{j,\bar{k}} \frac{\partial}{\partial z^j} \frac{\partial}{\partial \bar{z}^k} \big(\log \det(g_{p\bar{q}})\big) dz^j \wedge d\bar{z}^k. \end{aligned}$$

Note that this shows that ρ_ω is closed.

Ricci curvature

Moreover, if $\rho_{\tilde{\omega}}$ is the Ricci form of another Kähler metric (not necessarily in the same class!) then

$$\rho_\omega - \rho_{\tilde{\omega}} = i\partial\bar{\partial}\Big(\log \frac{\det(\tilde{g}_{p\bar{q}})}{\det(g_{p\bar{q}})}\Big).$$

We have that

$$\frac{\det(\tilde{g}_{p\bar{q}})}{\det(g_{p\bar{q}})} = \frac{\tilde{\omega}^n}{\omega^n},$$

which is a **globally** defined function. Here the notation $\frac{\tilde{\omega}^n}{\omega^n}$ means that this is the unique function f such that $\tilde{\omega}^n = f\omega^n$.
So

$$[\rho_\omega] = [\rho_{\tilde{\omega}}].$$

This class is

$$2\pi c_1(X) = -2\pi c_1(K_X),$$

since we can interpret the Ricci curvature as the (negative) of the induced curvature on K_X.

Kähler–Einstein metrics

Returning to the canonical metric question, we could then ask for ω to be a:

Kähler-Einstein metric:

$$\lambda\omega = \rho_\omega,$$

for some $\lambda \in \mathbb{R}$. But then $\lambda\Omega = 2\pi c_1(X)$. So $c_1(X)$ is either trivial (when $\lambda = 0$) or has a definite sign, and apart from the Calabi-Yau case $c_1(X) = 0$, the class Ω is then even pre-determined up to scale.

So in order to get a condition which makes sense on *any* Kähler manifold and any class, we need to relax the curvature condition.

Scalar curvature

Can contract ρ_ω with the metric to get the scalar curvature

$$S(\omega) = \Lambda_\omega(\rho_\omega).$$

We then seek a *constant scalar curvature (cscK)* metric in Ω:

$$S(\omega_\phi) = \text{constant}.$$

This question makes sense in any Kähler class. So the initial canonical metric question, for us, in higher dimensions becomes: **Given (X, Ω) does there exist an $\omega \in \Omega$ with constant scalar curvature?**

Fact: The constant is predetermined by the class.

Obstructions to the existence of cscK metrics

Futaki's obstruction to cscK metrics

The aim of this talk is to discuss perturbation problems for the cscK equation. As a way of introducing some of the computations involved in this, we first show that there is an obstruction to the existence of cscK metrics due to Prof. Futaki. These come from holomorphic vector fields on X. This will also help motivative the study of extremal Kähler metrics, a more general type of canonical metric.

Futaki's obstruction to cscK metrics

Definition 1.
A function h on X is a *holomorphy potential (wrt ω)* if $\nabla_{\omega}^{1,0} h$ is a holomorphic vector field, i.e.

$$\mathcal{D}_{\omega}(h) := \bar{\partial}(\nabla^{1,0} h) = 0.$$

It is a result of LeBrun-Simanca that the holomorphic vector fields that admit a holomorphy potential are precisely the ones with a zero somewhere. So having a holomorphy potential does not depend on the Kähler metric chosen. We will now show something weaker, namely that having a potential is independent of the metric chosen in a fixed class.

Futaki's obstruction to cscK metrics

Lemma 2.
If ν is a holomorphic vector field on X with potential h with respect to ω, then $h + \nu(\phi)$ is a holomorphy potential with respect to ω_ϕ.

Proof of Lemma 2

Locally, we have $\nu = \sum_j \nu^j \frac{\partial}{\partial z^j}$, with $\frac{\partial}{\partial \bar{z}^k}\nu^j = 0$ for all j, k, since ν is holomorphic. Moreover, since h is a holomorphy potential with respect to ω, we have

$$\nu^j = \sum_k g^{j\bar{k}} \frac{\partial h}{\partial \bar{z}^k}$$

using

$$g(\nu, \frac{\partial}{\partial \bar{z}^k}) = g(\nabla h, \frac{\partial}{\partial \bar{z}^k}) = dh(\frac{\partial}{\partial \bar{z}^k}).$$

Proof of Lemma 2

$$\nu^j = \sum_k g^{j\bar{k}} \frac{\partial h}{\partial \bar{z}^k}$$

Now, if $g_{\phi,j\bar{p}}$ denotes the components of g_ϕ,

$$\begin{aligned}
\sum_j g_{\phi,j\bar{p}}\nu^j &= \sum_j (g_{j\bar{p}} + \phi_{j\bar{p}})\nu^j \\
&= \sum_j \left(g_{j\bar{p}} (\sum_k g^{j\bar{k}} \partial_{\bar{k}} h) + \phi_{j\bar{p}}\nu^j \right) \\
&= \sum_k \delta_{p,k} \partial_{\bar{k}} h + \sum_j \phi_{j\bar{p}}\nu^j \\
&= \partial_{\bar{p}}(h) + \partial_{\bar{p}}\big(\sum_j \phi_j \nu^j\big) \\
&= \partial_{\bar{p}}(h + \nu(\phi)).
\end{aligned}$$

Proof of Lemma 2

$$\sum_j g_{\phi,j\bar{p}} \nu^j = \partial_{\bar{p}}(h + \nu(\phi)).$$

Applying the inverse to $g_{\phi,j\bar{p}}$ we then get that

$$\nu^j = \sum_p g_\phi^{j\bar{p}} \partial_{\bar{p}}(h + \nu(\phi)),$$

and so

$$\nu = \nabla^{1,0}_{\omega_\phi}(h + \nu(\phi)),$$

as required.

Futaki's obstruction to cscK metrics

Definition 3.
Let $\nu \in \mathfrak{h}$, the space of holomorphic vector fields with a potential. The *Futaki invariant* is the functional

$$F_\omega : \mathfrak{h} \to \mathbb{C}$$

defined by

$$F_\omega(v) = \int_X h_\omega(S(\omega) - \hat{S}_\Omega)\omega^n,$$

where h_ω is a holomorphy potential for ν with respect to ω.

Futaki's obstruction to cscK metrics

$$F_\omega(v) = \int_X h_\omega(S(\omega) - \hat{S}_\Omega)\omega^n$$

A-priori, this depends on ω. Futaki showed that this is not the case.

Theorem 4 (Futaki).

$$F_\omega = F_\Omega$$

is independent of the metric ω chosen in Ω.

Corollary 5.

If there is $\tilde{\omega} \in \Omega$ which is cscK, then $F_\Omega = 0$.

Proof of Theorem 4

We want to show that

$$F_{\omega_\phi}(\nu) = \int_X h_\phi(S(\omega_\phi) - \hat{S}_\Omega)\omega_\phi^n$$

equals $F_\omega(\nu)$, where we are using the shorthand h_ϕ for the changed potential, given by Lemma 2. It suffices to show that for any ω and for all $\phi \in C^\infty(X)$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(F_{\omega_{t\phi}}(\nu)\right) = 0.$$

From Lemma 2, we know $h_{t\phi} = h + \nu(t\phi)$, and so

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(h_{t\phi}\right) = \nu(\phi).$$

Proof of Theorem 4

Also,

$$\begin{aligned}\left(\omega_{t\phi}\right)^n &= \left(\omega + ti\partial\bar{\partial}\phi\right)^n \\ &= \omega^n + nti\partial\bar{\partial}\phi \wedge \omega^{n-1} + \cdots,\end{aligned}$$

and so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(\left(\omega_{t\phi}\right)^n\right) &= ni\partial\bar{\partial}\phi \wedge \omega^{n-1} \\ &= \Lambda_\omega(i\partial\bar{\partial}\phi)\omega^n \\ &= \Delta_\omega(\phi)\omega^n,\end{aligned}$$

the Laplacian applied to ϕ.

Proof of Theorem 4

We also need to know the linearisation of the scalar curvature. Since

$$S(\omega_{t\phi})\omega_{t\phi}^n = n\rho_{t\phi} \wedge \omega_{t\phi}^{n-1}, \tag{1}$$

this can be computed from the linearisations of $\omega_{t\phi}^k$, $k = n, n-1$, and from the linearisation of the Ricci curvature.

Proof of Theorem 4

To see the change in the linearisation of the Ricci curvature, we use that

$$\rho_{\omega_{t\phi}} - \rho_\omega = -i\partial\bar{\partial}\Big(\log \frac{\omega_{t\phi}^n}{\omega^n}\Big),$$

to get that

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\Big(\rho_{t\phi}\Big) = -i\partial\bar{\partial}\big(\Delta(\phi)\big).$$

Continuing using the identity (1),

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\big(S(\omega_{t\phi})\big)\omega^n = -\left(\Delta^2(\phi) + \langle\rho, i\partial\bar{\partial}\phi\rangle\right)\omega^n,$$

where we have used the identity
$n(n-1)\alpha\wedge\beta\wedge\omega^{n-2} = \left(\Lambda_\omega(\alpha)\Lambda_\omega(\beta) - \langle\alpha,\beta\rangle\right)\omega^n.$

Proof of Theorem 4

Finally one obtains

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\Big(S(\omega_{t\phi})\Big) = -\mathcal{D}^*_\omega\mathcal{D}_\omega(\phi) + \langle\nabla^{1,0}\phi, \nabla^{1,0}S(\omega)\rangle_\omega,$$

where we recall $\mathcal{D}_\omega(\phi) = \bar{\partial}(\nabla^{1,0}_\omega\phi)$. Using that $S(\omega)$ is real, we can conjugate and also obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\Big(S(\omega_{t\phi})\Big) = -\overline{\mathcal{D}^*_\omega\mathcal{D}_\omega(\phi)} + \langle\nabla^{1,0}S(\omega), \nabla^{1,0}\phi\rangle_\omega.$$

Proof of Theorem 4

With all of this in place, we can now compute the derivative of the Futaki invariant in the direction of ϕ:

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(F_{\omega_{t\phi}}(\nu)\right) = \int_X \bigg(v(\phi)(S(\omega) - \hat{S}_\Omega) + h(S(\omega) - \hat{S}_\Omega)\Delta(\phi) - h\overline{\mathcal{D}^*_\omega \mathcal{D}_\omega(\phi)} + h\langle \nabla^{1,0} S(\omega), \nabla^{1,0}\phi\rangle \bigg)\omega^n,$$

Using integration by parts, we see that

$$\int_X \left(h\overline{\mathcal{D}^*_\omega \mathcal{D}_\omega(\phi)}\right)\omega^n = 0,$$

since h is a holomorphy potential.

Proof of Theorem 4

Using the identity

$$\int_X f\Delta(\phi)\omega^n = -\int_X \langle \nabla^{1,0} f, \nabla^{1,0}\phi\rangle \omega^n$$

applied to $f = h(S(\omega) - \hat{S}_\Omega)$ we see that

$$\int_X \Big(h(S(\omega) - \hat{S}_\Omega)\Delta(\phi) \Big)\omega^n,$$

cancels with

$$\int_X \Big(v(\phi)(S(\omega) - \hat{S}_\Omega) + h\langle \nabla^{1,0} S(\omega), \nabla^{1,0}\phi\rangle \Big)\omega^n.$$

Hence the derivative vanishes, as required.

K-stability and the YTD conjecture

There are more obstructions to the existence of cscK/extremal metrics, coming from degenerating X to other manifolds, or even singular varieties. K-stability is such a notion. It involves:

- a class of a degenerations, called test configurations;
- a numerical invariant, the Donaldson–Futaki invariant, associated to each test configuration.

K-stability then asks for the Donaldson–Futaki invariant to have a particular sign.

Test configurations

The class of degenerations are defined as follows. Let (X, L) be a polarised manifold or variety. A *test configuration* $(\mathcal{X}, \mathcal{L})$ of exponent r is a normal polarised variety with a map $\pi : \mathcal{X} \to \mathbb{C}$ such that

- $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ is a flat family;
- $(\mathcal{X}, \mathcal{L})$ admits a $\mathbb{C}^*$-action, such that π is equivariant with respect to the standard action on $\mathbb{C}$;
- all non-zero fibres $(\mathcal{X}_t, \mathcal{L}_t)$ are isomorphic to (X, L^r).

From a test configuration, over $\mathbb{C}$ as above, one can define a compactified test configuration over $\mathbb{P}^1$. Simply glue with $X \times \mathbb{C}$ over $\mathbb{C}^*$. Sometimes we will think of test configurations in this way.

The Donaldson–Futaki invariant

Next, we associate a number to a test configuration. Originally, these were defined in terms of asymptotic expansions of the dimension of certain vector spaces and weights of actions on these vector spaces. However, they have since been shown by Odaka and Wang to be given by an intersection number on the total space of the compactified test configuration, and it is this formula we give.

The Donaldson–Futaki invariant

So, let $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1$ be a test configuration over $\mathbb{P}^1$, of exponent r. Let $K_{\mathcal{X}/\mathbb{P}^1} = K_{\mathcal{X}} - \pi^* K_{\mathbb{P}^1}$ denote the relative canonical bundle, and let $\mu(X, L) = \frac{-K_X.L^{n-1}}{L^n}$ denote the slope of (X, L). Then the *Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{L})$ is given by

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{n}{n+1} \mu(X, L^r) \mathcal{L}^{n+1} + K_{\mathcal{X}/\mathbb{P}^1}.\mathcal{L}^n.$$

When $(\mathcal{X}, \mathcal{L})$ is a *product test configuration* $X \times \mathbb{C}$ with an action on the X-component generated by a holomorphic vector field on X, one recovers the classical Futaki invariant of the vector field.

K-stability

Definition 6.

The polarised normal variety (X, L) is

- *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations;
- *K-stable* if further $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is a trivial test configuration;
- *K-polystable* if (X, L) is semistable and $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is a product test configuration.

The YTD conjecture

A central conjecture in the field is the Yau–Tian–Donaldson (YTD) conjecture:

Conjecture (Yau–Tian–Donaldson).

A polarised projective manifold (X, L) admits a cscK metric in $c_1(L)$ if and only if it is K-stable.

Extremal Kähler metrics

Extremal metrics

There is an even more general notion of canonical metric, due to Calabi. It generalises cscK metrics in the case when the reduced automorphism group is non-zero, which is precisely when the Futaki invariant provides an obstruction to the existence of cscK metrics. Extremal Kähler metrics are defined as the critical points of the energy functional associated to the scalar curvature operator. In other words, as critical points of the functional

$$\mathcal{K}_\omega \to \mathbb{R}$$

given by

$$\phi \mapsto \int_X S(\omega_\phi)^2 \omega_\phi^n.$$

Extremal metrics

When X is compact (which we always assume), this is equivalent to

$$\mathcal{D}_{\omega_\phi}(S(\omega_\phi)) = 0,$$

where

$$\mathcal{D}_{\omega_\phi} = \bar{\partial}(\nabla^{1,0}_{\omega_\phi}(f)).$$

Definition 7 (Calabi).

A Kähler metric ω on a compact Kähler manifold is *extremal* if

$$\mathcal{D}_\omega(S(\omega)) = 0.$$

When are extremal metrics cscK?

The Futaki invariant captures precisely when an extremal metric is cscK.

Proposition 1.

Suppose $\omega \in \Omega$ is an extremal Kähler metric. Then ω is cscK if and only the Futaki invariant F_Ω vanishes.

The point is that for a non-cscK extremal metric, one can pick $h = S(\omega) - \hat{S}$ as the holomorphy potential, and the corresponding vector field has positive Futaki invariant.

Rephrasing the equation

Since the scalar curvature map is a fourth order operator, the extremal equation

$$\mathcal{D}_\omega(S(\omega)) = 0$$

is order six. However, we can also view the equation as

$$S(\omega) \in \overline{\mathfrak{h}}_\omega,$$

where $\overline{\mathfrak{h}}_\omega$ is the finite dimensional space of potentials for real holomorphic vector fields on X. In this way we can view the equation as a fourth order equation.

Rephrasing the equation

Note that the space $\overline{\mathfrak{h}}_{\omega}$ depends on ω. Indeed, given a holomorphic vector field with zeros, we know the potential for it will change with the Kähler metric (Lemma 2). We are thus trying to hit a moving target. When doing analysis it is convenient to work with fixed spaces of functions, and we will now look into how we can do this for the extremal equation, when phrasing the equation in this way.

Rephrasing the equation

First note that while the holomorphic vector fields do not depend on ω, the ones with a real potential, i.e. the real holomorphic vector fields, do. However, if one fixes a maximal torus T of the reduced automorphism group of X and only works with torus-invariant metrics and potentials, then this stops being the case. Thus we will always assume from now (without mention) that we are have fixed a maximal torus and are working with torus-invariant metrics and functions.

Rephrasing the equation

Having done this, we are assured that when we use Lemma 2, a real potential remains real after changing the metric by a torus-invariant function. If h is a real holomorphy potential on X with respect to ω, we then have that

$$h + \frac{1}{2}\langle \nabla h, \nabla \phi \rangle$$

is a real holomorphy potential on X with respect to ω_ϕ. Thus to solve the extremal equation on X, we want to find a (torus-invariant) function $\phi : X \to \mathbb{R}$ and a holomorphy potential h such that

$$S(\omega_\phi) = h + \frac{1}{2}\langle \nabla h, \nabla \phi \rangle.$$

The extremal equation

In this way, we can view the equation as an equation between fixed spaces. If $\overline{\mathfrak{h}}$ denotes the space of potentials for real holomorphic vector fields with respect to ω, we seek the root of the map

$$C^{k+4,\alpha} \times \overline{\mathfrak{h}} \to C^{k,\alpha}$$

given by

$$(\phi, h) \mapsto S(\omega_\phi) - h - \frac{1}{2}\langle \nabla h, \nabla \phi \rangle.$$

In fact, one can show that the h we are trying to hit is predetermined, once the maximal torus is fixed (this is due to Futaki–Mabuchi).

Perturbation problems for extremal Kähler metrics

Perturbation problems

We now come to the main point of the talk, where we discuss the overall strategy involved in perturbation problems for the extremal equation.

These problems are of the following type.

- Start with (X, Ω) such that Ω admits an extremal metric;
- Perturb the problem: consider $(\tilde{X}, \tilde{\Omega})$ that is "close" to (X, Ω);
- This involves a parameter, say ε. So, we have a family $(\tilde{X}_\varepsilon, \tilde{\Omega}_\varepsilon)$;
- As $\varepsilon \to 0$, we get closer and closer to our original (X, Ω), where we know there is a cscK or extremal metric.

Often $\tilde{X}_\varepsilon$ does not depend on $\varepsilon > 0$, but may be different from X.

Sample problems

- LeBrun–Simanca openness:

$$(\tilde{X}_\varepsilon, \tilde{\Omega}_\varepsilon) = (X, \Omega + \varepsilon A).$$

- Arezzo–Pacard type blow up:

$$(\tilde{X}_\varepsilon, \tilde{\Omega}_\varepsilon) = (\mathrm{Bl}_p\, X, \pi^*\Omega - \varepsilon[E]).$$

- Fibrations (Fine, Hong, ...):

$$(\tilde{X}_\varepsilon, \tilde{\Omega}_\varepsilon) = (X, \varepsilon\Omega + \pi^*\Omega_B),$$

where X is the total space of a fibration $\pi : X \to B$ where the fibres X_b admit cscK metrics in $\Omega_{|X_b}$, and Ω_B is a Kähler class on B.

Overall strategy

Back to the general picture: One then uses the extremal metric on X in Ω to construct an approximately extremal metric $\omega_\varepsilon \in \tilde{\Omega}_\varepsilon$. The goal is then to show via some contraction mapping principle that this can be perturbed to a genuine solution of the extremal equation, at least when the parameter ε is very close to 0.

Overall strategy

This often relies on a *quantitative* inverse/implicit function theorem. We know from the inverse function theorem that if the linearisation of an operator $\mathcal{N}$ is invertible, then we can hit everything near $\mathcal{N}(0)$. However, in our ε-dependent situation we have a one parameter family $\mathcal{N}_\varepsilon$ of operators and while $\mathcal{N}_\varepsilon(0) \to 0$ as $\varepsilon \to 0$, it may be that the neighbourhood of $\mathcal{N}_\varepsilon(0)$ that we can hit also shrinks with ε. So in order to guarantee that we can solve the extremal equation when ε is sufficiently small, we need to show that the approximate solutions get better and better at a sufficiently fast rate. This will not really be an issue in the first of the problems mentioned above, but will feature on both of the latter two.

Overall strategy

This rate is determined by bounds on the inverse of the linearised operator. Recall that the linearisation of the scalar curvature operator is

$$-L_\omega + \frac{1}{2}\langle \nabla S(\omega), \nabla(\cdot)\rangle.$$

In particular, if $S(\omega)$ is constant, this linearised operator is the Lichnerowicz operator

$$L_\omega = -\mathcal{D}^*_\omega \mathcal{D}_\omega(\cdot),$$

and so the mapping properties of this operator is key.
Note that if X does not admit holomorphic vector fields with zeros, the cokernel of this operator is the constants. But if X does admit holomorphic vector fields, then this cokernel increases. This usually complicates these type of problems when there are holomorphic vector fields present.

The LeBrun–Simanca openness theorem, a sample problem

The LeBrun–Simanca theorem

We now discuss one such problem.

Theorem 8 (LeBrun–Simanca).

The set of Kähler classes that admit an extremal metric is an open subset of the Kähler cone.

In other words, we wish to show that if Ω is a Kähler class that admits extremal metrics, then there exists an open subset

$$U \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$$

about the origin such that for all $A \in U$, there exists an extremal Kähler metric in the class $\Omega + A$.

Case 1: no holomorphic vector fields

Start with (X, Ω) admitting a cscK metric ω, and assume there are no holomorphic vector fields. Let $A = [\alpha] \in H^{1,1}(X, \mathbb{R})$.
Goal: solve the cscK equation in the class

$$\Omega + \varepsilon A.$$

So we want to solve

$$S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi) = c_{\varepsilon,A},$$

where $c_{\varepsilon,A} = \hat{S}_{\Omega+\varepsilon A}$ is a topological constant approaching $\hat{S}_\Omega$ as $\varepsilon \to 0$. This is a perturbation problem of the type we have been discussing where we are not changing the manifold, only the class.

Constructing an approximate solution

The first step is then to construct a good approximate solution. We are simply going to use $\omega_\varepsilon = \omega + \varepsilon\alpha$. It is Kähler for all ε sufficiently close to 0. Moreover, we can certainly ensure $S(\omega_\varepsilon)$ is bounded independently of ε when ε is sufficiently small.

Constructing an approximate solution

To see that this indeed is going to give better and better approximate solutions, we again use the identity

$$S(\omega_\varepsilon)\omega_\varepsilon^n = n\rho_{\omega_\varepsilon} \wedge \omega_\varepsilon^{n-1}.$$

We have that

$$\begin{aligned}\omega_\varepsilon^n &= (\omega + \varepsilon\alpha)^n \\ &= \omega^n + n\varepsilon\alpha \wedge \omega^{n-1} + \cdots \\ &= \big(1 + \varepsilon\Lambda_\omega(\alpha) + \cdots\big)\omega^n.\end{aligned}$$

Constructing an approximate solution

We similarly get

$$
\begin{aligned}
&\rho_{\omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \\
&= \big(\rho_\omega - \varepsilon i\partial\bar{\partial}(\Lambda_\omega(\alpha)) + \cdots\big) \wedge \big(\omega^{n-1} + (n-1)\varepsilon\alpha \wedge \omega^{n-2}\big) \\
&= \rho_\omega \wedge \omega^{n-1} + \varepsilon\big(-i\partial\bar{\partial}(\Lambda_\omega(\alpha))\omega^{n-1} + (n-1)\rho_\omega \wedge \alpha \wedge \omega^{n-2}\big) + O(\varepsilon^2).
\end{aligned}
$$

Constructing an approximate solution

So

$$\begin{aligned}
&S(\omega_\varepsilon)\omega^n \\
=&S(\omega)\omega^n \\
&+\varepsilon\big(-S(\omega_\varepsilon)\Lambda_\omega(\alpha)\omega^n - ni\partial\bar{\partial}(\Lambda_\omega(\alpha))\omega^{n-1} + n(n-1)\rho_\omega\wedge\alpha\wedge\omega^{n-2}\big) \\
&+O(\varepsilon^2) \\
=&S(\omega)\omega^n \\
&+\varepsilon\big(-S(\omega_\varepsilon)\Lambda_\omega(\alpha) - \Delta(\Lambda_\omega(\alpha)) - \langle\rho_\omega,\alpha\rangle + S(\omega)\Lambda_\omega(\alpha)\big)\omega^n \\
&+O(\varepsilon^2).
\end{aligned}$$

Constructing an approximate solution

This allows us to deduce that there is a constant C such that if ε is sufficiently small, then

$$\|S(\omega_\varepsilon) - S(\omega)\| \leq C\varepsilon.$$

This is in C^0, but one could apply similar ideas to get bounds for the derivatives too. So ω_ε is an approximate solution to the cscK equation, approaching a cscK metric at $O(\varepsilon)$.

Mapping properties of the linearisation

We now wish to perturb this into a genuine solution. That is, we wish to show that we can obtain a zero of the map

$$\mathcal{N}_\varepsilon : C^\infty(X) \times \mathbb{R} \to C^\infty(X)$$

given by

$$(\phi, c) \mapsto S(\omega_\varepsilon + i\partial\bar{\partial}\phi) - S(\omega) - c.$$

Our previous computation is then saying $\mathcal{N}_\varepsilon(0,0) = O(\varepsilon)$ is an approximate root of this map.

Mapping properties of the linearisation

To perturb, we need to look at the linearisation, which we saw was given by

$$P_\varepsilon = -L_{\omega_\varepsilon}(\cdot) + \frac{1}{2}\langle \nabla S(\omega_\varepsilon), \nabla \cdot \rangle.$$

This is a complicated ε-dependent operator. But since ω_ε approaches ω and is approximately cscK, it is well approximated by $P = -L_\omega$, a self-adjoint operator that we know well. This means that

$$\|P_\varepsilon - P\| \leq C'\varepsilon$$

in operator norm.

Mapping properties of the linearisation

Under our assumption that there are no holomorphic vector fields, P is an isomorphism on $C_0^\infty(X) \times \mathbb{R}$, where the subscript denotes functions of average 0. More appropriately, it is an isomorphism on the Hölder spaces

$$C_0^{k+4,\alpha}(X) \times \mathbb{R} \to C^{k,\alpha}(X),$$

since we need to work in Banach spaces (elliptic regularity theory will then allow us to show that a solution to the cscK equation is in fact smooth).

Mapping properties of the linearisation

Since this is an open condition, it follows that P_ε is an isomorphism too, when ε is sufficiently small, and that we even get a bound

$$\frac{1}{C''}\|Q\| \leq \|Q_\varepsilon\| \leq C''\|Q\|$$

for the inverse Q_ε of P_ε, for some constant C'' independent of ε.

Applying the Inverse Function Theorem

Now we can complete the proof of the existence of the cscK metric. Since the inverse of the linearisation of $\mathcal{N}_\varepsilon$ has a definite bound, the inverse function theorem implies that there is a neighbourhood of $\mathcal{N}_\varepsilon(0,0)$ of definite size, independently of ε, that can be hit with $\mathcal{N}_\varepsilon$. In particular, since $\mathcal{N}_\varepsilon(0,0) \to 0$ as $\varepsilon \to 0$, there is a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists $(\phi_\varepsilon, c_\varepsilon)$ such that $\mathcal{N}_\varepsilon(\phi_\varepsilon, c_\varepsilon) = 0$, i.e.

$$S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi_\varepsilon) = S(\omega) + c_\varepsilon,$$

a constant.

Case 2: there are holomorphic vector fields

If X admits holomorphy potentials, then we have a larger cokernel than the constants for L_ω and so a larger cokernel for the linearised operator. The cokernel is then holomorphy potentials with respect to ω. If we start with a cscK metric, this gives an obstruction to solving the cscK equation in nearby classes (in fact, this is the Futaki invariant we discussed earlier).

Case 2: there are holomorphic vector fields

However, by incorporating the change in holomorphy potentials with α and ϕ, one can always solve for

$$S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi)$$

being a holomorphy potential with respect to $\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi$, i.e. we can find an extremal metric in nearby classes. Restricting to starting with a cscK metric is then unnecessary, and we assume that our initial ω is an extremal metric on X.

The cscK condition is not necessarily preserved

We will show that that if one starts with an extremal metric, one can perturb to another extremal metric in nearby classes. Note that if one starts with a cscK metric, it may or may not be that the extremal metric obtained in the perturbed class is cscK. Whether or not this is the case is exactly captured by the Futaki invariant. For example, $X = \mathrm{Bl}_{p_1,p_2,p_3}\,\mathbb{P}^2$ admits a Kähler-Einstein metric. For this class, i.e. $\Omega = c_1(-K_X)$, the volume of the three exceptional divisors are the same. All nearby Kähler classes admit extremal metrics, but they are cscK if and only if the volume of the exceptional divisors remain the same as it is only when the exceptional divisors have the same volume that the Futaki invariant vanishes.

Proof – case 2

Suppose we start with having an extremal metric ω in some class Ω. If h_ν is the potential for a holomorphic vector field ν on X with respect to ω and h_ν^α is a holomorphy potential for ν with respect to $\omega + \alpha$ (assume α is small enough that this is Kähler), then

$$h_{\nu,\varepsilon} = h_\nu + \varepsilon \left(h_\nu^\alpha - h_\nu \right)$$

is a holomorphy potential for ν with respect to $\omega_\varepsilon = \omega + \varepsilon\alpha$. If we perturb to $\omega_\varepsilon + i\partial\bar{\partial}\phi$, the new holomorphy potential is

$$h_{\nu,\varepsilon} + \frac{1}{2}\langle \nu, \nabla\phi \rangle.$$

Proof – case 2

The equation we wish to solve is then

$$S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi_\varepsilon) = h_{\nu,\varepsilon} + \frac{1}{2}\langle \nu, \nabla\phi\rangle + c_\varepsilon,$$

to obtain an extremal metric. I.e. we wish to find a zero of the map

$$\Phi : C^{4,\alpha}(X) \times \mathfrak{h} \times \mathbb{R} \to C^{0,\alpha}(X),$$

given by

$$(\phi, \nu, c) \mapsto S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi_\varepsilon) - h_{\nu,\varepsilon} - \frac{1}{2}\langle \nu, \nabla\phi\rangle - c.$$

Proof – case 2

$$(\phi, \nu, c) \mapsto S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi_\varepsilon) - h_{\nu,\varepsilon} - \frac{1}{2}\langle \nu, \nabla\phi\rangle - c.$$

The linearisation of this operator is surjective. We remove the cokernel with the $\mathfrak{h} \times \mathbb{R}$ factor. The term $\frac{1}{2}\langle \nu, \nabla\phi\rangle$ precisely kills off a bad looking term coming from the linearisation of the scalar curvature at a non-cscK metric. So by following the steps of case 1, we can use a quantitative *implicit* function theorem to obtain the required extremal metric.

Final remark

Remark:.
Technically, we have only showed openness about Ω in the one-parameter family $\Omega + \varepsilon A$ of Kähler classes on X. However, using the finite-dimensionality of the space in which Ω lies ($H^{1,1}(X) \cap H^2(X, \mathbb{R})$),it is straightforward to see that we get the required openness statement.

Thank you!