Effective joint equidistribution of primitive rational points on expanding horospheres

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Contents

Farey sequences

- The Farey sequence of order $q \geq 1$, denoted by F_q, is the sequence of completely reduced fractions between 0 and 1, which when in lowest terms have denominators less than or equal to q, arranged in order of increasing size. For example,

$$F_6 = \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1} \right\}.$$

- Let $S_q = F_q \setminus F_{q-1} = \{ \frac{r}{q} \ : \ r \in \mathbb{Z} \cap [1, q], \ \gcd(r, q) = 1 \}$ be the set of rationals in $[0, 1]$ with denominator $= q$.
  $\# S_q = \varphi(q) \gg q / \log \log q$, where φ is Euler's totient function.
  For example, if $q = p$ is a prime, then

$$S_p = \left\{ \frac{r}{p} \ : \ 1 \leq r \leq p - 1 \right\} \quad \text{and} \quad \# S_p = p - 1.$$

- Equadistribution theorem: for any fixed $0 \leq \alpha < \beta \leq 1$,

$$\lim_{q \to \infty} \frac{\# S_q \cap [\alpha, \beta]}{\# S_q} = \beta - \alpha.$$

Equadistribution of reduced fractions

Claim: For any smooth function F on $\mathbb{T} \cong [0,1)$, we have

$$\boxed{\frac{1}{\varphi(q)} \sum_{\substack{r=1\\(r,q)=1}}^{q} F(r/q) = \int_0^1 F(x)\mathrm{d}x + O_F(q^{-1+\varepsilon}).}$$

Proof: By Fourier analysis, we get $F(y) = \sum_{n\in\mathbb{Z}} \left(\int_0^1 F(x)e(-nx)\mathrm{d}x\right) e(ny)$, where $e(z) = e^{2\pi i z}$. Hence

$$\frac{1}{\varphi(q)} \sum_{\substack{r=1\\(r,q)=1}}^{q} F(r/q) = \int_0^1 F(x)\mathrm{d}x + \frac{1}{\varphi(q)} \sum_{n\neq 0} \left(\int_0^1 F(x)e(-nx)\mathrm{d}x\right) \sum_{\substack{r=1\\(r,q)=1}}^{q} e\left(\frac{nr}{q}\right).$$

By the integral we know that we can truncate n-sum at $|n| \ll q^{\varepsilon}$ with a negligible error term. Now we consider the Ramanujan sums

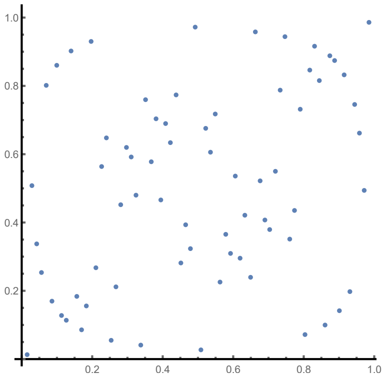
$$R(n;q) = \sum_{\substack{r=1\\(r,q)=1}}^{q} e\left(\frac{nr}{q}\right) = \sum_{d|(n,q)} d\mu(q/d) \ll n^{1+\varepsilon} \ll q^{\varepsilon}.$$

This proves our claim.

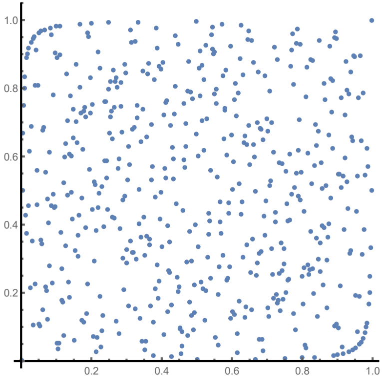
Joint distribution

For any $r \in \mathbb{Z} \cap [1, q]$, $\gcd(r, q) = 1$, we define $\bar{r} \in \mathbb{Z} \cap [1, q]$ to be the inverse of r modulo q, that is, $r\bar{r} \equiv 1 \pmod{q}$.

A natural question is the joint distribution of $(r/q, \bar{r}/q)$ in $[0, 1]^2$.

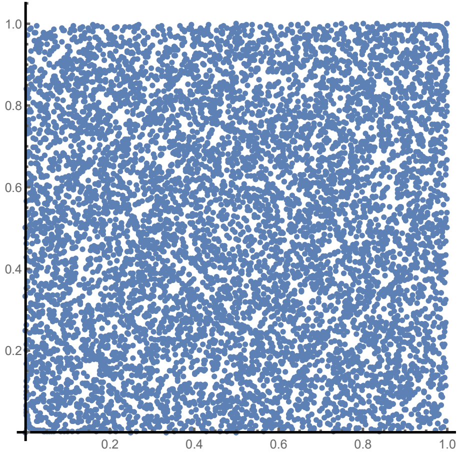


$q = p_{20} = 71$



$q = p_{100} = 541$

Joint distribution



$q = p_{1000} = 7919$

Joint equidistribution

Theorem (Marklof 2010 and Einsiedler–Mozes–Shah–Shapira 2016)

Let $F : \mathbb{T}^2 \to \mathbb{R}$ be a smooth function, then for any positive $c < 1/2$, we have

$$\boxed{\frac{1}{\varphi(q)} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} F(r/q, \bar{r}/q) = \int_0^1 \int_0^1 F(x,y)\mathrm{d}x\mathrm{d}y + O_F(q^{-c}).}$$

Sketch of proof: By Fourier expansion, we have

$$\begin{aligned}
\Sigma &= \frac{1}{\varphi(q)} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} F(r/q, \bar{r}/q) \\
&= \frac{1}{\varphi(q)} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} e\left(\frac{mr + n\bar{r}}{q}\right) \int_0^1 \int_0^1 F(x,y) e(-mx - ny)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Joint equidistribution

Rearranging the sums, we get

$$\Sigma = \frac{1}{\varphi(q)} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} S(m,n;q) \int_0^1 \int_0^1 F(x,y) e(-mx-ny) \mathrm{d}x\mathrm{d}y,$$

where $S(m,n;q) := \sum_{\substack{r=1\\(r,q)=1}}^{q} e\left(\frac{mr+n\bar{r}}{q}\right)$ is the Kloosterman sum.

- The case $m=n=0$. Note that $S(0,0;q)=\varphi(q)$, hence we get the main term $\int_0^1 \int_0^1 F(x,y)\mathrm{d}x\mathrm{d}y$.
- The case $m=0$ and $n\neq 0$. Note that $S(0,n;q)$ is the Ramanujan's sum, for which we have $|S(0,n;q)| \leq \gcd(n,q)$. Hence the contribution is $\ll q^{-1+\varepsilon}$.
- The case $m\neq 0$ and $n=0$ is the same as above.
- The case $m\neq 0$ and $n\neq 0$. By the Riemann Hypothesis for curves over a finite field (due to André Weil 1948), we have

$$S(m,n;q) \ll q^{1/2+\varepsilon} \gcd(m,n,q)^{1/2}.$$

  Hence the contribution is $\ll q^{-1/2+\varepsilon}$.

Group theory interpretation

Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and define

$$H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\} \subset G.$$

Denote by $\mathrm{d}x$ the H-invariant Haar probability measure on $\Gamma\backslash\Gamma H \cong \mathbb{R}/\mathbb{Z}$.

For $r \in \mathbb{Z} \cap [1, q]$, $\gcd(r, q) = 1$, we have

$$\Gamma \begin{pmatrix} 1 & 0 \\ r/q & 1 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} = \Gamma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \Gamma\backslash\Gamma H,$$

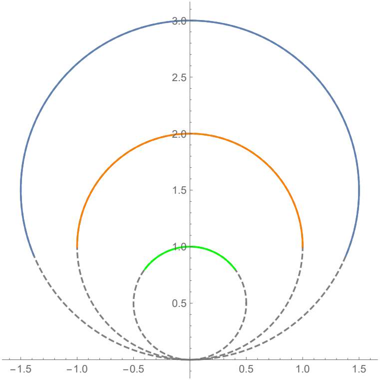
where $x = \bar{r}/q$, as $\begin{pmatrix} q & 0 \\ r & q^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\bar{r}/q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & -\bar{r} \\ r & -r\bar{r}/q + 1/q \end{pmatrix} \in \Gamma.$

Let $f : \Gamma\backslash\Gamma H \times \mathbb{T} \to \mathbb{R}$ be a smooth function, then

$$\boxed{\frac{1}{\varphi(q)} \sum_{\substack{r=1 \\ (r,q)=1}}^{q} f\left(\Gamma \begin{pmatrix} 1 & 0 \\ r/q & 1 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, r/q \right) = \int_{\Gamma\backslash\Gamma H \times \mathbb{T}} f \mathrm{d}x \mathrm{d}y + O_f(q^{-1/2+\varepsilon}).}$$

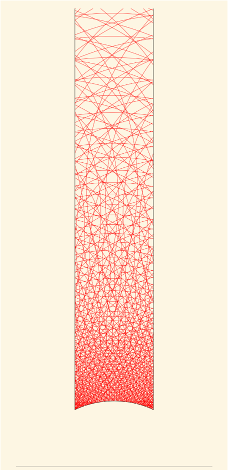
Expanding horocycles

$$\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} i = \frac{vq^4}{v^2q^4+1} + \frac{q^2}{v^2q^4+1} i$$



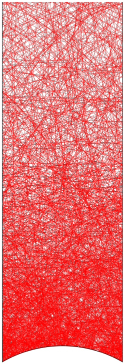
Expanding horocycles with $-1/2 \leq v \leq 1/2$ and $q = 1,\ \sqrt{2},\ \sqrt{3}$.

Horocycle equidistribution

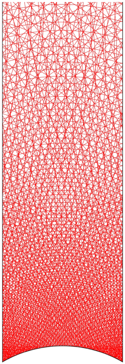


From Constantin Kogler's webpage

Flows on the Modular Surface



Geodesic Flow



Horocycle Flow

Notations

Define, for $d \geq 1$ and $q \geq 1$,

$$\mathcal{R}_q = \{\mathbf{r} \in (\mathbb{Z} \cap [1,q])^d \ : \ \gcd(\mathbf{r}, q) = 1\}$$

and

$$D(q) = \mathrm{diag}(q^{\frac{1}{d}}, \ldots, q^{\frac{1}{d}}, q^{-1}) \in \mathrm{SL}_{d+1}(\mathbb{R}).$$

Let $G = \mathrm{SL}_{d+1}(\mathbb{R})$ and $\Gamma = \mathrm{SL}_{d+1}(\mathbb{Z})$ and define

$$H = \left\{ \begin{pmatrix} A & \mathbf{v} \\ {}^{\mathrm{t}}\mathbf{0} & 1 \end{pmatrix} \ : \ A \in \mathrm{SL}_d(\mathbb{R}), \mathbf{v} \in \mathbb{R}^d \right\} \subset G.$$

Denote by μ_H the H-invariant Haar probability measure on $\Gamma \backslash \Gamma H$. Finally, for $\mathbf{x} \in \mathbb{R}^d$, define

$$n_+(\mathbf{x}) = \begin{pmatrix} I_d & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{x} & 1 \end{pmatrix} \in G.$$

We have

$$\Gamma n_+(q^{-1}\mathbf{r}) D(q) \in \Gamma \backslash \Gamma H.$$

Main theorem

Let $\mathrm{C}_b^k(\Gamma\backslash\Gamma H \times \mathbb{T}^d)$ be the space of k times continuously differentiable functions with all derivatives bounded.

Theorem (El-Baz–H–Lee 2022+)

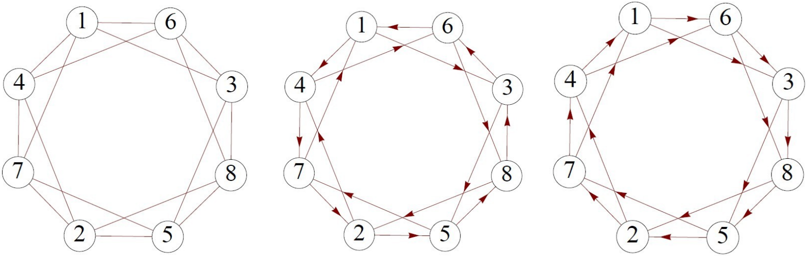
For every $d \geq 3$, every $\varepsilon > 0$ and every integer $k \geq 2d^2 - d + 1$, there exists a constant $c > 0$ such that for every function $f \in \mathrm{C}_b^k(\Gamma\backslash\Gamma H \times \mathbb{T}^d)$ and every $q \in \mathbb{Z}_{\geq 1}$,

$$\left| \frac{1}{\#\mathcal{R}_q} \sum_{\mathbf{r} \in \mathcal{R}_q} f\left(\Gamma n_+\left(\frac{1}{q}\mathbf{r}\right) D(q), \frac{1}{q}\mathbf{r}\right) - \int_{\Gamma\backslash\Gamma H \times \mathbb{T}^d} f \,\mathrm{d}\mu_H \mathrm{d}\mathbf{x} \right| \leq c \|f\|_{\mathrm{C}_b^k} q^{-\frac{1}{2} + \frac{d^2(2k-2d+1)}{2k^2} + \varepsilon}.$$

Remarks

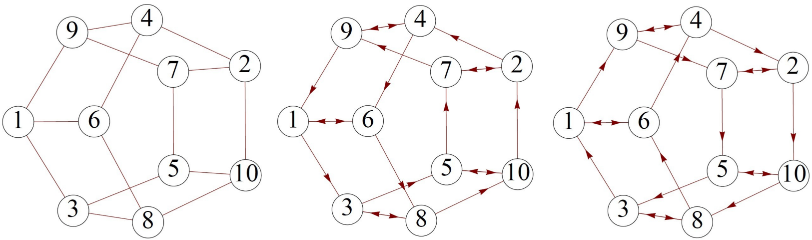
- We have "continuous version", e.g. Nimish Shah (1996).
- "Average version" over q is due to Jens Marklof (2010) and Han Li (2015).
- Our result is an effective version of a result of Manfred Einsiedler, Shahar Mozes, Nimish Shah, and Uri Shapira (2016).
- Andreas Strömbergsson (2015) proved an effective Ratner equidistribution result for the affine special linear group $\mathrm{ASL}_2(\mathbb{R})$.
- The case $d = 1$ is "easy", which is already known at least by Marklof (2010) and Einsiedler, Mozes, Shah, and Shapira (2016).
- The case $d = 2$ is due to Min Lee and Jens Marklof (2018).

Circulant graphs



The 4-regular circulant graph $C_8(2,3)$ and the circulant digraphs $C_8^+(2,3)$, $C_8^+(2,5)$. The corresponding diameters are 2, 3 and 4, respectively.

Circulant graphs



The 3-regular circulant graph $C_{10}(2,5)$ and the circulant digraphs $C_{10}^{+}(2,5)$, $C_{10}^{+}(5,8)$. The corresponding diameters are 3, 5 and 5, respectively.

Undirected circulant graphs

Let us fix an integer vector $\mathbf{a} = (a_1, \ldots, a_d)$ with distinct positive coefficients $0 < a_1 < \ldots < a_d \leq \frac{q}{2}$.

We construct a graph $C_q(\mathbf{a})$ with q vertices $1, 2, \ldots, q$, by connecting vertex i and j whenever $|i - j| \equiv a_h \bmod q$ for some $h \in \{1, \ldots, d\}$.

If $a_d < \frac{q}{2}$, then $C_q(\mathbf{a})$ is $2d$-regular, i.e., every vertex has precisely $2d$ neighbours.
If $a_d = \frac{q}{2}$, then $C_q(\mathbf{a})$ is $(2d-1)$-regular.

$C_q(\mathbf{a})$ is connected if and only if $\gcd(a_1, \ldots, a_d, q) = 1$.

Diameter of a circulant graph

We endow our circulant graph with a (quasi-)metric by stipulating that the edge from i to $j \equiv i + a_h \bmod q$ has length 1. We denote the corresponding metric graphs by $C_q(\mathbf{a})$ itself.

The distance $d(i,j)$ between two vertices is the length of the shortest path from i to j. The diameter is the maximal distance between any pair of vertices,

$$\mathrm{diam}\ C_q(\mathbf{a}) = \max_{i,j} d(i,j).$$

Amir and Gurel-Gurevich (2010) conjectured the existence of a limiting distribution, as $q \to +\infty$, for $\frac{\mathrm{diam}\ C_q(\mathbf{a})}{q^{1/d}}$.

Limiting distribution

Marklof and Strömbergsson (2013) proved the limiting distribution of the diameters with averaging over q.

The existence of this limiting distribution is a consequence of the main theorem Einsiedler, Mozes, Shah, and Shapira (2016).

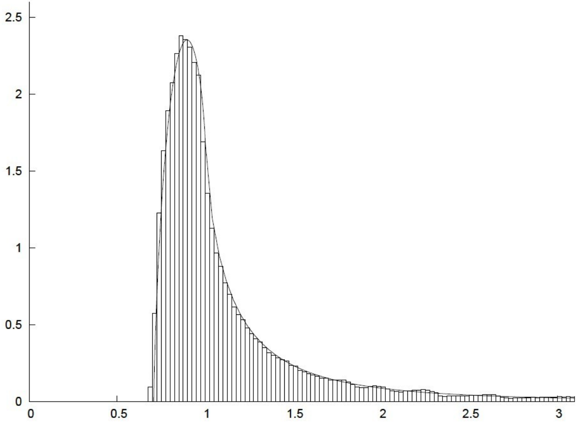
Corollary (El-Baz–H–Lee 2022+)

For every $d \geq 2$, there exists a continuous non-increasing function $\Psi_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Psi_d(0) = 1$ and a constant $\eta_d > 0$ such that for every $R \geq 0$, we have

$$\mathrm{Prob}\left(\frac{\mathrm{diam}\, C_q(\mathbf{a})}{q^{1/d}} \geq R\right) = \Psi_d(R) + O\left(q^{-\eta_d}\right),$$

where the implicit constant depends on R.

Limiting distribution



Distribution of diameters diam $C_q(\mathbf{a})$ with average over q for circulant graphs with $d = 2$.

Proof Sketch of Main Theorem: Congruence equations

For $\mathbf{r} \in \mathcal{R}_q$, there exist $A \in \mathrm{SL}_d(\mathbb{R})$ and $\mathbf{x} \in \mathbb{R}^d$ such that

$$\Gamma n_+(q^{-1}\mathbf{r})D(q) = \Gamma \begin{pmatrix} A & \mathbf{x} \\ {}^{\mathrm{t}}\mathbf{0} & 1 \end{pmatrix} \in \Gamma\backslash\Gamma H.$$

This is equivalent to the existence of $A \in \mathrm{SL}_d(\mathbb{R})$ and $\mathbf{x} \in \mathbb{R}^d$, uniquely determined modulo $\Gamma = \mathrm{SL}_{d+1}(\mathbb{Z})$, satisfying

$$\begin{aligned}\begin{pmatrix} A & \mathbf{x} \\ {}^{\mathrm{t}}\mathbf{0} & 1 \end{pmatrix}(n_+(q^{-1}\mathbf{r})D(q))^{-1} &= \begin{pmatrix} A & \mathbf{x} \\ {}^{\mathrm{t}}\mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} q^{-\frac{1}{d}} I_d & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & q \end{pmatrix}\begin{pmatrix} I_d & \mathbf{0} \\ -q^{-1}\,{}^{\mathrm{t}}\mathbf{r} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{q^{1-\frac{1}{d}}A - q\mathbf{x}\,{}^{\mathrm{t}}\mathbf{r}}{q} & q\mathbf{x} \\ -\,{}^{\mathrm{t}}\mathbf{r} & q \end{pmatrix} \in \Gamma.\end{aligned}$$

Let $\mathbf{s} = q\mathbf{x}$ and $B = q^{\frac{d-1}{d}}A$. By the above relation,

$$\mathbf{s} \in \mathbb{Z}^d, \quad \frac{1}{q}(B - \mathbf{s}\,{}^{\mathrm{t}}\mathbf{r}) \in \mathrm{M}_d(\mathbb{Z}) \quad \text{and } \det(B) = q^{d-1}\det(A) = q^{d-1}.$$

So

$$B \in \mathrm{M}_d(\mathbb{Z}) \quad \text{and} \quad B \equiv \mathbf{s}\,{}^{\mathrm{t}}\mathbf{r} \pmod{q}.$$

Parametrizing $\mathcal{R}_q$

For a positive integer q, we also define the following congruence subgroup:

$$\Gamma_{0,d}(q) = \left\{ \gamma \in \mathrm{SL}_d(\mathbb{Z}) \ : \ \gamma \equiv \begin{pmatrix} * & * \\ {}^{\mathrm{t}}\mathbf{0} & u \end{pmatrix} \pmod q, \ \gcd(u,q) = 1 \right\}.$$

We can parametrise $\mathcal{R}_q$ in terms of $\Gamma_{0,d}(q)\backslash \mathrm{SL}_d(\mathbb{Z})$ and $(\mathbb{Z}/q\mathbb{Z})^\times$.
Let $\mathcal{B}_q$ be a set of representatives for $\Gamma_{0,d}(q)\backslash \mathrm{SL}_d(\mathbb{Z})$.

Lemma (El-Baz–H–Lee 2022+)

We have

$$\mathcal{R}_q = \left\{ \mathbf{r} \equiv {}^{\mathrm{t}}\gamma \begin{pmatrix} \mathbf{0} \\ u \end{pmatrix} \pmod q \ : \ \gamma \in \mathcal{B}_q, \ u \in (\mathbb{Z}/q\mathbb{Z})^\times \right\}.$$

Let $B_0 = \left(\begin{smallmatrix} qI_{d-1} & \\ & 1 \end{smallmatrix}\right)$, for any $\gamma \in \Gamma_{0,d}(q)\backslash \mathrm{SL}_d(\mathbb{Z})$ and $u \in (\mathbb{Z}/q\mathbb{Z})^\times$, if we set

$$\begin{aligned} \mathbf{r} &\equiv u\,{}^{\mathrm{t}}\gamma \mathbf{e}_d \pmod q, \\ \mathbf{s} &= \overline{u}\mathbf{e}_d, \ u\overline{u} \equiv 1 \pmod q, \\ B &= B_0\gamma, \end{aligned}$$

then $\mathbf{r} \in \mathcal{R}_q$, $\det(B) = q^{d-1}$ and $B \equiv \mathbf{s}\,{}^{\mathrm{t}}\mathbf{r} \pmod q$.

Connection to Kloosterman sums

We have

$$\begin{aligned}
&\frac{1}{\#\mathcal{R}_q}\sum_{\mathbf{r}\in\mathcal{R}_q} f\left(\Gamma n_+\left(\frac{1}{q}\mathbf{r}\right)D(q), \frac{1}{q}\mathbf{r}\right)\\
&\quad= \frac{1}{\#\mathcal{R}_q}\sum_{\gamma\in\mathcal{B}_q}\sum_{u\in(\mathbb{Z}/q\mathbb{Z})^\times} f\left(\begin{pmatrix} q^{-1+\frac{1}{d}}B_0\gamma & q^{-1}\overline{u}\mathbf{e}_d \\ {}^{\mathrm{t}}\mathbf{0} & 1\end{pmatrix}, q^{-1}u\,{}^{\mathrm{t}}\gamma\mathbf{e}_d\right)\\
&\quad= \frac{1}{\#\mathcal{R}_q}\sum_{\gamma\in\mathcal{B}_q}\sum_{u\in(\mathbb{Z}/q\mathbb{Z})^\times}\sum_{\mathbf{n}\in\mathbb{Z}^d} \widehat{f}_{\mathbf{n}}\left(\begin{pmatrix} q^{-1+\frac{1}{d}}B_0\gamma & q^{-1}\overline{u}\mathbf{e}_d \\ {}^{\mathrm{t}}\mathbf{0} & 1\end{pmatrix}\right) e\left(\frac{{}^{\mathrm{t}}\mathbf{n}u\,{}^{\mathrm{t}}\gamma\mathbf{e}_d}{q}\right),
\end{aligned}$$

where

$$\widehat{f}_{\mathbf{n}}(g) = \int_{(\mathbb{R}/\mathbb{Z})^d} f(g,\mathbf{t})e\left(-{}^{\mathrm{t}}\mathbf{n}\mathbf{t}\right)\,\mathrm{d}\mathbf{t}.$$

Connection to Kloosterman sums

For $A \in \mathrm{SL}_d(\mathbb{R})$ and $\mathbf{y} \in \mathbb{R}^d$, let $F_{\mathbf{n}}(A, \mathbf{y}) = \widehat{f}_{\mathbf{n}}\left(\begin{pmatrix} A & \mathbf{y} \\ {}^{\mathrm{t}}\mathbf{0} & 1 \end{pmatrix}\right)$.

We have

$$
\begin{aligned}
&\frac{1}{\#\mathcal{R}_q} \sum_{\mathbf{r} \in \mathcal{R}_q} f\left(\Gamma n_+\left(\frac{1}{q}\mathbf{r}\right) D(q), \frac{1}{q}\mathbf{r}\right) \\
&= \frac{1}{\#\mathcal{R}_q} \sum_{\gamma \in \mathcal{B}_q} \sum_{\mathbf{n} \in \mathbb{Z}^d} \sum_{\mathbf{m} \in \mathbb{Z}^d} \widehat{F_{\mathbf{n}}}\left(q^{-1+\frac{1}{d}} B_0 \gamma, \mathbf{m}\right) \sum_{u \in (\mathbb{Z}/q\mathbb{Z})^\times} e\left(\frac{{}^{\mathrm{t}}\mathbf{m}\bar{u}\mathbf{e}_d + {}^{\mathrm{t}}\mathbf{n}u\,{}^{\mathrm{t}}\gamma\mathbf{e}_d}{q}\right) \\
&= \frac{1}{\#\mathcal{R}_q} \sum_{\gamma \in \mathcal{B}_q} \sum_{\mathbf{n}} \sum_{\mathbf{m} \in \mathbb{Z}^d} \widehat{F_{\mathbf{n}}}\left(q^{-1+\frac{1}{d}} B_0 \gamma, \mathbf{m}\right) S(m_d, {}^{\mathrm{t}}\mathbf{n}({}^{\mathrm{t}}\gamma\mathbf{e}_d); q),
\end{aligned}
$$

where $\widehat{F_{\mathbf{n}}}(A, \mathbf{m}) = \int_{(\mathbb{R}/\mathbb{Z})^d} F_{\mathbf{n}}(A, \mathbf{t})\, e\left(-{}^{\mathrm{t}}\mathbf{m}\mathbf{t}\right) \mathrm{d}\mathbf{t}$.

Now by the Riemann Hypothesis for curves over a finite field, equidistribution of Hecke points (due to Laurent Clozel, Hee Oh, and Emmanuel Ullmo 2001), and counting techniques, we can prove our main theorem. $\square$

Thank you for your attention!