PART I. $SL_2(\mathbb{F}_q)$

Chapter 1. Structure

Chapter 2. Harish-Chandra induction

Chapter 3. Introduction to ℓ-adic cohomology

TODAY Chapter 4. Deligne-Lusztig theory for $SL_2(\mathbb{F}_q)$

Chapter 4. Deligne-Lusztig induction

4. A. Drinfeld curve

$$Y = \{(x,y) \in \mathbb{A}^2(\mathbb{F}) \mid xy^q - yx^q = 1\}$$

$$\begin{cases} G \text{ acts on the left} \\ \mu_{q+1} \simeq T' \text{ acts on the right} \end{cases}$$

$H^i_c(Y)$ is a (KG, KT')-bimodule

4. B. Definition

$$\mathcal{R}'_i : KT'\text{-mod} \longrightarrow KG\text{-mod}$$
$$M \longmapsto H^i_c(Y) \otimes_{KT'} M$$

$$R' = \sum_{i \geq 0} (-1)^{i+1} R'_i : \mathrm{Class}(T') \to \mathrm{Class}(G)$$

$$R'(\mathrm{Irr}\, T') \subset \mathbb{Z}\,\mathrm{Irr}(G)$$

(Deligne-Lusztig induction)

Let $\theta \in \mathrm{Irr}\, T' = \mathrm{Hom}_{gp}(T', K^\times)$

$$R'(\theta) = \chi_{H^1_c(Y) \otimes_{KT'} E_\theta} - \chi_{H^2_c(Y) \otimes_{KT'} E_\theta}$$

$$= -\frac{1}{q+1} \sum_{\xi \in \mu_{q+1}} \mathrm{Tr}^*_Y(g,\xi)\, \theta(\xi^{-1})$$

where E_θ is a KT'-module (of dim. 1) affording the character θ.

$$\boxed{\langle R(\alpha), R'(\theta)\rangle = 0}$$

Harish-Chandra induction.

$$\mathcal{R} : KT\text{-mod} \longrightarrow KG\text{-mod}$$
$$KT\text{-mod} \cong K\mu_{q-1}\text{-mod}$$

$$R : \mathrm{Class}(T) \longrightarrow \mathrm{Class}(G)$$

Prop. 2.4. $R(\alpha)(1) = (q+1)\,\alpha(1)$

Mackey formula 2.5. $\langle R(\alpha), R(\beta)\rangle_G = \langle \alpha, \beta\rangle_T + \langle \alpha, {}^{s}\beta\rangle_T$

Theo. 2.6. Let $\alpha, \beta \in \mathrm{Irr}(T) = \mathrm{Hom}(T, K^\times)$.

(a) $R(\alpha) = R(\alpha^{-1})$

(b) $\langle R(\alpha), R(\beta)\rangle = 0$ if $\beta \neq \alpha^{\pm 1}$

(c) if $\alpha^2 \neq 1$, then $R(\alpha) \in \mathrm{Irr}(G)$

Example 2.7. $R(1_T) = 1_G + \mathrm{St}$

Example 2.8. $R(\alpha_0) = R(\alpha_0)^+ + R(\alpha_0)^-$

(2.9) $$R(\chi_{KT}) = \underbrace{1_G}_{\text{dim. }1} + \underbrace{\mathrm{St}}_{\text{dim. }q} + \underbrace{R(\alpha_0)^+}_{\text{dim. }\frac{q+1}{2}} + \underbrace{R(\alpha_0)^-}_{\text{dim. }\frac{q+1}{2}} + 2\sum_{\alpha \in (\mathrm{Irr}\,T \setminus \{1,\alpha_0\})/\mathrm{inv}} \underbrace{R(\alpha)}_{\text{dim } q+1}$$

$\frac{q+5}{2}$ distinct irreducible characters.

Deligne - Lusztig induction

$$\mathcal{R}'_i : KT'\text{-mod} \longrightarrow KG\text{-mod}$$
$$KT'\text{-mod} \cong K\mu_{q+1}\text{-mod}$$

$$R' = R'_1 - R'_2 : \mathrm{Class}(T') \longrightarrow \mathrm{Class}(G)$$

$R'(\theta)(1) = (q-1)\,\theta(1)$

Mackey formula. $\langle R'(\theta), R'(\eta)\rangle_G = \langle \theta, \eta\rangle_{T'} + \langle \theta, {}^{s'}\eta\rangle_{T'}$

Theo. Let $\theta, \eta \in \mathrm{Irr}(T') = \mathrm{Hom}(T', K^\times)$

(a) $R'(\theta) = R'(\theta^{-1})$

(b) $\langle R'(\theta), R'(\eta)\rangle = 0$ if $\theta \neq \eta^{\pm 1}$

(c) If $\theta^2 \neq 1$, then $R'(\theta) \in \mathrm{Irr}\,G$

Example. $R'(1_{T'}) = -1_G + \mathrm{St}$

Example. $R'(\theta_0) = R'(\theta_0)^+ + R'(\theta_0)^-$

$$R'(\chi_{KT'}) = -\underbrace{1_G}_{\text{dim. }1} + \underbrace{\mathrm{St}}_{\text{dim. }q} + \underbrace{R'(\theta_0)^+}_{\text{dim. }\frac{q-1}{2}} + \underbrace{R'(\theta_0)^-}_{\text{dim }\frac{q-1}{2}} + 2\sum_{\theta \in (\mathrm{Irr}\,T \setminus \{1,\theta_0\})/\mathrm{inv}} \underbrace{R'(\theta)}_{\text{dim. }q-1}$$

$\frac{q+3}{2}$ distinct NEW irreducible characters

4. C. Quotients of varieties by finite groups

Let X be an algebraic variety
Let Γ be a finite group acting on X.

Theorem 4.3 (Serre). The set of orbits X/Γ is endowed with a unique structure of variety such that:

(1) The quotient morphism $\pi : X \to X/\Gamma$ is a morphism of varieties

(2) For any morphism $\varphi : X \longrightarrow X'$ which is constant on Γ-orbits, there exists a unique morphism $\bar{\varphi} : X/\Gamma \to X'$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & X/\Gamma \arrow[d, dashed, "\bar{\varphi}"] \\ & X' \end{tikzcd}$$

is commutative.

Sketch of the proof (Serre, "Algebraic groups and class fields", III. 3. 12). Since X is quasi-projective, there exists an open covering $(U_i)_{i\in I}$ of X where each U_i is affine and Γ-stable.

$\Rightarrow$ By glueing, we may assume that X is affine.

In this case, take X/Γ to be the affine variety such that
$$\mathcal{O}(X/\Gamma) = \mathcal{O}(X)^\Gamma \quad \text{(folklore)}. \blacksquare$$

Remark 4.4. $\dim X/\Gamma = \dim X$.

Proposition 4.5. Let $\varphi : X \longrightarrow X'$ be a morphism such that:

(1) X and X' are smooth

(2) φ is surjective

(3) $\varphi(x_1) = \varphi(x_2) \iff x_1$ and x_2 belong to the same Γ-orbit

(4) There exists $x_0 \in X$ such that $d_{x_0}\varphi : T_{x_0}X \longrightarrow T_{\varphi(x_0)}X'$ is surjective.

Then the map $\bar{\varphi} : X/\Gamma \longrightarrow X'$ induced by φ (and theo. 4.3 (2)) is an isomorphism.

Borel, "Linear algebraic groups", Prop. 6.6

Remark. In char. 0, (4) is not necessary. $\blacksquare$

4.D. Some quotients of $Y = \{(x,y)\in \mathbb{A}^2 \mid xy^q - yx^q = 1\}$

Let

$$\begin{aligned} \gamma : \quad Y &\longrightarrow \mathbb{A}^1(\mathbb{F}) \\ (x,y) &\longmapsto xy^{q^2} - yx^{q^2} \end{aligned}$$

$$\begin{aligned} \nu : \quad Y &\longrightarrow \mathbb{A}^1(\mathbb{F}) \setminus \{0\} \\ (x,y) &\longmapsto y \end{aligned}$$

$$\begin{aligned} \pi : \quad Y &\longrightarrow \mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q) \\ (x,y) &\longmapsto [x,y] \end{aligned}$$

- γ is constant on G-orbits
- ν is constant on U-orbits $\quad U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$
- π is constant on μ_{q+1}-orbits

Theorem 4.6.

(a) The map $\bar{\gamma} : Y/G \longrightarrow \mathbb{A}^1(\mathbb{F})$ is an isomorphism

(b) The map $\bar{\nu} : Y/U \longrightarrow \mathbb{A}^1(\mathbb{F}) \setminus \{0\}$ is an isomorphism

(c) The map $\bar{\pi} : Y/\mu_{q+1} \longrightarrow \mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q)$ is an isomorphism.

Proof of (a). We apply Proposition 4.5 ((b) and (c) are similar but easier: exercise)

$\gamma : Y \longrightarrow \mathbb{A}^1(\mathbb{F})$

$(x,y) \longmapsto x y^{q^2} - y x^{q^2}$

(1) Y and $\mathbb{A}^1(\mathbb{F})$ are smooth.

Let $a \in \mathbb{F}$: $\gamma^{-1}(a) = \{ (x,y) \in \mathbb{F}^\times \times \mathbb{F}^\times \mid xy^q - yx^q = 1 \text{ and } xy^{q^2} - yx^{q^2} = a \}$

We only need to prove that $1 \leqslant |\gamma^{-1}(a)| \leqslant |G| = q(q^2-1)$ (because G acts freely: 4.1(b))

Let $(z,t) = (x, \frac{y}{x})$

then $\gamma^{-1}(a) \overset{\sim}{\longleftrightarrow} \mathcal{E}_a = \{ (z,t) \in \mathbb{F}^\times \times \mathbb{F}^\times \mid t^q - t = z^{-1-q} \text{ and } t^{q^2} - t = a z^{-1-q^2} \}$

Notice that $t^{q^2} - t = (t^q - t)^q + (t^q - t)$

so $\mathcal{E}_a = \{ (z,t) \in \mathbb{F}^\times \times \mathbb{F}^\times \mid \underbrace{t^q - t = z^{-1-q}} \text{ and } \underbrace{z^{q^2-1} - a z^{q-1} + 1 = 0} \}$

Once z is fixed:
- at least one solution
- at most q solutions.

- at least one solution
- at most q^2-1 solutions

$\Rightarrow 1 \leqslant |\mathcal{E}_a| = |\gamma^{-1}(a)| \leqslant q(q^2-1)$. This shows (2) and (3).

- Let $v = (x_0, y_0) \in Y$: $T_v(Y) = \{ (x,y) \in \mathbb{F}^2 \mid y_0^q x - x_0^q y = 0 \}$

$$d_v \gamma (x,y) = y_0^{q^2} x - x_0^{q^2} y$$

$$\operatorname{Ker} d_v\gamma (x,y) = \{ (x,y) \in \mathbb{F}^2 \mid y_0^q x - x_0^q y = y_0^{q^2} x - x_0^{q^2} y = 0 \}$$

the determinant of this system is $(x_0 y_0^q - y_0 x_0^q)^q = 1 \neq 0$.

$\Rightarrow$ (4). ∎

4. E. Curiosities.

① **Automorphisms**: Recall that $\tilde{G} = GL_2(\mathbb{F}_q)$.

Let $\mathcal{G} = \{ (g, \xi) \in \tilde{G} \times \mathbb{F}_{q^2}^\times \mid \det(g)\, \xi^{1+q} = 1 \} \rhd \Delta = \{ (aI_2, a^{-1}) \mid a \in \mathbb{F}_q^\times \}$

$\mathcal{G} \supset G \times \mu_{q+1}$

Then • $\mathcal{G}$ acts on Y

• $\Delta = \mathrm{Ker}(\text{action of } \mathcal{G} \text{ on } Y)$

So $\mathcal{G}/\Delta \hookrightarrow \mathrm{Aut}(Y)$

Let $\overline{Y} = \{ [x, y, z] \in \mathbb{P}^2(\mathbb{F}) \mid xy^q - yx^q = z^{q+1} \}$: compactification of Y

$\mathcal{G}/\Delta \hookrightarrow \mathrm{Aut}(\overline{Y})$. $\Rightarrow |\mathrm{Aut}(\overline{Y})| \geqslant |\mathcal{G}| = q(q-1)(q+1)^2$

<u>Easy fact</u>. $\overline{Y}$ is smooth, irreducible, projective of degree $q+1$.

<u>Consequence</u>. $g(\overline{Y}) = \dfrac{q(q-1)}{2}$ (genus)

<u>Theorem (Hurwitz)</u>. If $\mathcal{C}$ is a smooth irreducible projective curve of genus $g \geqslant 2$ over $\mathbb{C}$, then $|\mathrm{Aut}(\mathcal{C})| \leqslant 84(g-1)$.

Note that $|\mathrm{Aut}(\overline{Y})| > 84(g(\overline{Y}) - 1) = 42(q-2)(q+1)$ if $q \geqslant 7$.

② Unramified Galois coverings. X irreducible and X' irreducible

Definition. An unramified (étale) Galois covering of X is a morphism $\pi : X' \longrightarrow X$ such that the group $\Gamma = \mathrm{Aut}(X'/X) = \{ \gamma \in \mathrm{Aut}(X') \mid \pi \circ \gamma = \pi \}$ is finite, acts freely on X and is such that $X'/_\Gamma \simeq X$; Γ is called the Galois group of π.

Example in char. 0. The map

$$\mathbb{C}^\times \longrightarrow \mathbb{C}^\times, \quad t \longmapsto t^n$$

is an unramified Galois covering with group μ_n.

Example in char. p. The map

$$\pi : \mathbb{A}^1(\mathbb{F}) \longrightarrow \mathbb{A}^1(\mathbb{F}), \quad t \longmapsto t^q - t$$

is an unramified Galois covering with group $\mathbb{F}_q^+$ ($\pi(t+a) = \pi(t)$ if $a \in \mathbb{F}_q$).

Artin–Schreier covering.

Fact. There is no non-trivial unramified Galois covering of $\mathbb{A}^1(\mathbb{C})$ (because $\mathbb{C}$ is simply-connected).

Abhyankar's Lemma. There is no non-trivial unramified Galois covering of $\mathbb{A}^1(\mathbb{F})$ with Galois group of order prime to p.

[SGA1, Exposé XIII, 5.3]

Let Γ be a finite group.
We denote by $O^{p'}(\Gamma)$ the subgroup of Γ generated by its Sylow p-subgroups.
($\Gamma/_{O^{p'}(\Gamma)}$ is the largest quotient of Γ of order prime to p).

Corollary. If Γ is the Galois group of an unramified Galois covering of $\mathbb{A}^1(\mathbb{F})$, then $\Gamma = O^{p'}(\Gamma)$.

$$\left(X'/_\Gamma \simeq \mathbb{A}^1(\mathbb{F}) \Rightarrow \left(X'/_{O^{p'}(\Gamma)}\right) \Big/ \left(\Gamma/_{O^{p'}(\Gamma)}\right) \simeq \mathbb{A}^1(\mathbb{F}) \right)$$

Abhyankar's conjecture (1957)
Raynaud's Theorem (1994)
The converse holds!

Example. U is a Sylow p-subgroup of G and $G = \langle U, {}^g U \rangle = O^{p'}(G)$.

$$Y/_G \simeq \mathbb{A}^1(\mathbb{F}) \quad (!!)$$

$$\boxed{\langle R(\alpha), R'(\theta)\rangle = 0}$$

Harish-Chandra induction.

$\mathcal{R} : KT\text{-mod} \longrightarrow KG\text{-mod}$

$\quad\ \| $

$K\mu_{q-1}\text{-mod}$

$R : \mathrm{Class}(T) \longrightarrow \mathrm{Class}(G)$

Prop. 2.4. $R(\alpha)(1) = (q+1)\,\alpha(1)$

Mackey formula 2.5. $\langle R(\alpha), R(\beta)\rangle_G = \langle \alpha, \beta\rangle_T + \langle \alpha, {}^{s}\beta\rangle_T$

Theo. 2.6. Let $\alpha, \beta \in \mathrm{Irr}(T) = \mathrm{Hom}(T, K^\times)$.

(a) $R(\alpha) = R(\alpha^{-1})$

(b) $\langle R(\alpha), R(\beta)\rangle = 0$ if $\beta \neq \alpha^{\pm 1}$

(c) If $\alpha^2 \neq 1$, then $R(\alpha) \in \mathrm{Irr}(G)$

Example 2.7. $R(1_T) = 1_G + \mathrm{St}$

Example 2.8. $R(\alpha_0) = R(\alpha_0)^+ + R(\alpha_0)^-$

$$(2.9)\quad R(\chi_{KT}) = \underset{\text{dim. } 1}{1_G} + \underset{\text{dim. } q}{\mathrm{St}} + \underset{\text{dim. } \frac{q+1}{2}}{R(\alpha_0)^+} + \underset{\text{dim. } \frac{q+1}{2}}{R(\alpha_0)^-} + 2 \sum_{\alpha \in (\mathrm{Irr}\,T \smallsetminus \{1,\alpha_0\})/_{\mathrm{INV}}} \underset{\text{dim } q+1}{R(\alpha)}$$

$\frac{q+5}{2}$ distinct irreducible characters.

Deligne - Lusztig induction

$\mathcal{R}'_i : KT'\text{-mod} \longrightarrow KG\text{-mod}$

$\quad\ \| $

$K\mu_{q+1}\text{-mod}$

$R' = R'_1 - R'_2 : \mathrm{Class}(T') \longrightarrow \mathrm{Class}(G)$

$R'(\theta)(1) = (q-1)\,\theta(1)$

Mackey formula. $\langle R'(\theta), R'(\eta)\rangle_G = \langle \theta, \eta\rangle_{T'} + \langle \theta, {}^{s'}\eta\rangle$

Theo. Let $\theta, \eta \in \mathrm{Irr}(T') = \mathrm{Hom}(T', K^\times)$

(a) $R'(\theta) = R'(\theta^{-1})$

(b) $\langle R'(\theta), R'(\eta)\rangle = 0$ if $\theta \neq \eta^{\pm 1}$

(c) If $\theta^2 \neq 1$, then $R'(\theta) \in \mathrm{Irr}\, G$

Example. $R'(1_{T'}) = -1_G + \mathrm{St}$

Example. $R'(\theta_0) = R'(\theta_0)^+ + R'(\theta_0)^-$

$$R'(\chi_{KT'}) = -\underset{\text{dim. } 1}{1_G} + \underset{\text{dim. } q}{\mathrm{St}} + \underset{\text{dim. } \frac{q-1}{2}}{R'(\theta_0)^+} + \underset{\text{dim } \frac{q-1}{2}}{R'(\theta_0)^-} + 2 \sum_{\theta \in (\mathrm{Irr}\,T \smallsetminus \{1,\theta_0\})/_{\mathrm{INV}}} \underset{\text{dim. } q-1}{R'(\theta)}$$

$\frac{q+3}{2}$ distinct NEW irreducible characters

4. F. Proofs of all these results.

Proposition 4.7. $R'(1_{T'}) = -1_G + \mathrm{St}$

Proof.

$$\mathcal{R}'_1(K_{T'}) = H^1_c(Y) \otimes_{KT'} K_{T'}$$

$$= H^1_c(Y)^{T'} = H^1_c(Y)^{\mu_{q+1}}$$

$$= H^1_c(Y/\mu_{q+1}) \qquad (\text{theo. } 3.7(i))$$

$$= H^1_c(\mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q)) \qquad (4.4(c))$$

$$\underset{KG\text{-module}}{\simeq} \mathrm{St} \qquad (\text{example } 3.9)$$

$$\mathcal{R}'_2(K_{T'}) = H^2_c(Y) \otimes_{KT'} K_{T'}$$

$$= \dots$$

$$= H^2_c(\mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q))$$

$$\simeq K_G \qquad (\text{example } 3.9)$$

$$R'(1_{T'}) = R'_1(1_{T'}) - R'_2(1_{T'})$$

$$= \mathrm{St} - 1_G. \quad \blacksquare$$

Corollary 4.8. $R'(\theta)(1) = (q-1)\,\theta(1)$

Proof

$$R'(\theta)(1) = -\frac{1}{q+1} \sum_{\xi \in \mu_{q+1}} \mathrm{Tr}^*_Y(1,\xi)\, \theta(\xi^{-1})$$

If $\xi \neq 1$, then ξ has order prime to p,

so $\mathrm{Tr}^*_Y(1,\xi) = \mathrm{Tr}^*_{Y^\xi}(1) \qquad (\text{theo } 3.10)$

But $Y^\xi = \emptyset \Rightarrow \mathrm{Tr}^*_{Y^\xi}(1) = 0$

$$R'(\theta)(1) = -\frac{1}{q+1} \mathrm{Tr}^*_Y(1,1)\ \theta(1)$$

But for $\theta = 1_{T'}$, we get $R'(1_{T'})(1) = q-1$.

$$\Rightarrow -\frac{1}{q+1} \mathrm{Tr}^*_Y(1,1) = q-1. \quad \blacksquare$$

Proposition 4.9. If $(g,\xi) \in G \times \mu_{q+1}$, then $\mathrm{Tr}^*_Y(g,\xi) = \mathrm{Tr}^*_Y(g,\xi^{-1})$.

Proof. The Frob. endo F of Y induces an automorphism F_i on $H^i_c(Y)$ (cor. 3.12).
But, by remark 4.2,

$$F_i \circ (g,\xi) \circ F_i^{-1} = (g,\xi^{-1}). \quad \blacksquare$$

Corollary 4.10. $R'(\theta) = R'({}^{\sigma'}\theta) = R'(\theta)^*$

(where $R'(\theta)^*(g) = R'(\theta)(g^{-1})$)

Proof.

$$R'({}^{\sigma'}\theta)(g) = -\frac{1}{q+1}\sum_{\xi\in\mu_{q+1}} \mathrm{Tr}^*_Y(g,\xi)\,({}^{\sigma'}\theta)(\xi^{-1})$$

$$\overset{4.9\,\mathrm{II}}{=} -\frac{1}{q+1}\sum_{\xi\in\mu_{q+1}} \mathrm{Tr}^*_Y(g,\xi^{-1})\,\theta(\xi)$$

by definition

$$= R'(\theta)(g).$$

$$= \cdots = R'(\theta)(g^{-1})$$

$(\mathrm{Tr}^*_Y(g,\xi) = \mathrm{Tr}^*_Y(g^{-1},\xi^{-1}))$. ∎

For proving the Mackey formula, we will need the following two results.

Complement to Theorem 3.10.

(e) Let $f : X \to X$ and $f' : X' \to X'$ be Frob. endos or automorphisms of finite order. Then

$$\mathrm{Tr}^*_{X\times X'}(f\times f') = \mathrm{Tr}^*_X(f)\,\mathrm{Tr}^*_{X'}(f')$$

(consequence of Künneth formula 3.7(e))

Exercise 4.11. Let M be $K\Gamma$-module and let $\varphi \in \mathrm{End}_{K\Gamma}(M)$. Show that

$$\mathrm{Tr}(\varphi, M^\Gamma) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \mathrm{Tr}(\gamma\varphi, M)$$