Equivariant Lagrangian correspondences and applications

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April 6, 2024

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Supported by RGC grants of HK Government.

String theory on $Y^6 \times \mathbb{R}^{3,1}$

requires Y Kähler manifold w/ $\text{Ricci} = 0$.

Yau: Y compact Kähler w/ $c_1(Y) = 0$,

$\Rightarrow \exists$ Kähler metric w/ $\text{Ricci} = 0$.

called Calabi-Yau manifold.

Mirror Symmetry

Y, $\check{Y}$ mirror Calabi-Yau pair.

(Physics) String theory on Y $\xleftrightarrow{\text{dual}}$ String theory on $\check{Y}$

(Math) Symplectic Geometry Y $\longleftrightarrow$ $\check{Y}$ Complex Geometry

$\boxed{A}$ $\boxed{B}$

and vice versa.

Strominger-Yau-Zaslow conjecture

$Y \longleftrightarrow \check{Y}$

dual special Lagrangian tori fibrations.

Strominger-Yau-Zaslow conjecture

$Y \longleftrightarrow \check{Y}$ dual special Lagrangian tori fibrations.

In particular,

$\check{Y} =$ moduli space of (L, ∇)

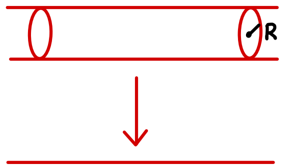
w/ $L \subset Y$ (special) Lagrangian torus

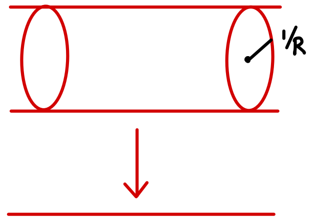
∇ flat $U(1)$-bundle $/ L$

Note: $\check{T} = \{$ flat $U(1)$-bundles $/ T \} / $isom.

Fiberwise Fourier-Mukai transf. between T and $\check{T}$ gives mirror transformations.

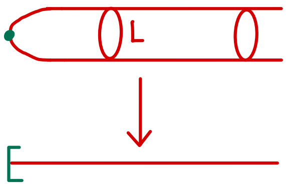
Eg. $Y = \mathbb{R} \times S^1_R = \mathbb{C}^\times \quad \longleftrightarrow \quad Y^\vee = \mathbb{R} \times S^1_{1/R} = \mathbb{C}^\times$

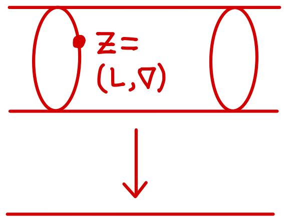




Eg. $Y = \mathbb{C} \quad \longleftrightarrow \quad Y^\vee = \mathbb{C}^\times \xrightarrow{W} \mathbb{C}$

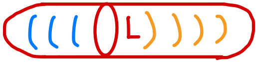
LG-model.

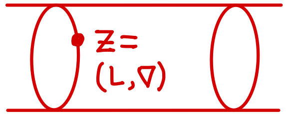




count holom. disks bound L weighed by holonomy of ∇ $\rightsquigarrow$ $W(z) = z$

Eg. $Y = \mathbb{CP}^1 \longleftrightarrow (\check{Y}, W) = \left(\mathbb{C}^\times, z + \frac{q}{z}\right)$ size of $\mathbb{CP}^1$





Alternative view: Quotient $\longleftrightarrow$ Fiber

$$\mathbb{C}^2 /\!/ S^1 = \mathbb{CP}^1 \longleftrightarrow \mathbb{C}^\times,\ W_{\mathbb{CP}^1} = z + \frac{q}{z}$$

fiber of F over q

$$\mathbb{C}^2 \longleftrightarrow \mathbb{C}^{\times 2},\ W_{\mathbb{C}^2} = z_1 + z_2$$

$$\begin{tikzcd} \mathbb{C}^{\times 2} \arrow[d, "F = z_1 z_2"] \\ \mathbb{C}^\times \end{tikzcd}$$

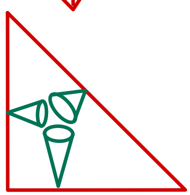
$\left(F = e^{\mu_{S^1}(L)} \cdot \mathrm{Hol}_{S^1}(\nabla)\right)$

($W_{\mathbb{C}^2} + \log F$, S^1-equivar. potential)

Similar for Fano toric $Y = \mathbb{C}^n /\!\!/ T^k$,

$(\check{Y}, W_{HF})$ called Hori-Vafa mirror.

$$Y = \mathbb{C}^{n+k} /\!\!/ T^k \longleftrightarrow \check{Y} = (\mathbb{C}^\times)^n$$



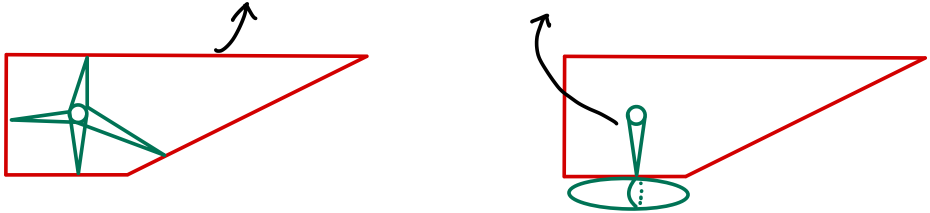
$$W_{HF} = z_1 + z_2 + \frac{q}{z_1 z_2}$$

$= \sum$ 1 term for each toric divisor in Y

If not Fano, need further corrections to W.

eg. Hirzebruch surface $Y = \mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1})$

$(\check{Y},\ W = W_{HF} + \text{extra term})$



(obstructed) holom. disk
(w/ bubble)

In general, corrections could be much more complicated.

Goal compact $G \curvearrowright (Y, \omega) \xrightarrow{\mu} \mathfrak{g}^*$

$X := Y /\!\!/_c G$ symplectic quotient (not necessary Fano)

Describe $(\check{X}, W_{\check{X}})$

S.C. Lau - L. - Y.L. Li : ✓ locally if $G = T$.

Equivariant theory, a review

$G \curvearrowright M$, $\quad C(M)^G = C(M/G) \leftarrow$ continuous functions

Equivar. cohomology $\quad H^\bullet_G(M) = H^\bullet(M/G)$ if $G \curvearrowright M$ free

$$H^\bullet_G(M) \triangleq H^\bullet(M \times_G E_G)$$

module over $H^\bullet(B_G) = (\mathrm{Sym}^\bullet \mathfrak{g}^*)^G$

$$\begin{tikzcd}
 & G \arrow[d] \arrow[rr, bend left] & & M \arrow[d] \\
{*} \sim & E_G \arrow[d] & & M \times_G E_G = M_G \arrow[d] \\
 & B_G & & B_G
\end{tikzcd}$$

Eg. $G = S^1$ $\quad G \longrightarrow E_G \longrightarrow B_G$

$$\equiv \quad S^1 \longrightarrow S^\infty \longrightarrow \mathbb{CP}^\infty$$

$$\equiv \quad \varinjlim_{N\to\infty} \left(S^1 \longrightarrow S^{2N+1} \longrightarrow \mathbb{CP}^N \right)$$

$\exists$ finite dim. approx.

$$H^\bullet(B_{S^1}) = \mathbb{C}[z]$$

In general, $\quad G \longrightarrow E_G \longrightarrow B_G$

$$\equiv \quad \varinjlim_{N\to\infty} \left(G \longrightarrow E_{G_N} \longrightarrow B_{G_N} \right)$$

Q.M. / 1d σ-model w/ SUSY $\overset{\text{Witten}}{\rightsquigarrow}$ $\mathcal{H} = H^\bullet(M)$

vector space

String/2d σ-model w/ SUSY $\rightsquigarrow$ category

A-model: $\mathcal{F}uk(Y,\omega)$

B-model: $D^bCoh(\check{Y},J)$

Homological Mirror Symmetry (Kontsevich)

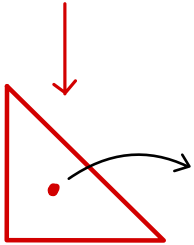
$\mathcal{F}uk(Y,\omega) \ni (L,\nabla)$, $L \subset Y$ Lagrangian submanifold.

∇ flat $U(1)$-connection on L

SYZ $\rightsquigarrow$ $\check{Y} = \{(L,\nabla)\}/\cong$ w/ quantum corrections

$$\check{Y} = \{(L,\nabla)\}/\cong \quad ?$$

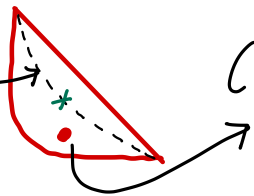
Eg. $Y = \mathbb{CP}^2$



$$\xrightarrow[\text{eg. Clifford torus}]{\text{toric fibers}} \quad (\mathbb{C}^\times)^2 \simeq MC_{weak}(T_{Cliff}) \subset \check{Y}$$

$T_{Cliff} = (S^1)^2 \subset \mathbb{C}^2 \subset \mathbb{CP}^2$

wall



Chekanov tori $\rightsquigarrow$ another chart $MC_{weak}(T_{Chek}) \subset \check{Y}$

Wall Crossing Formula $\longleftrightarrow$ gluing MC_{weak}'s charts $\rightsquigarrow \check{Y}$.

$L \overset{\text{Lagr.}}{\hookrightarrow} (Y, \omega)$ $\quad \overset{A}{\longleftrightarrow} \overset{B}{} \quad$ $\mathbb{C}^r \longrightarrow E \longrightarrow (\check{Y}, J)$

holom. VB $\quad F^{0,2} = 0.$

unobstructed $b \in CF^{odd}(L, L)$ $\quad$ or $\varphi \in \Omega^{0,1}(\check{Y}, E^* \otimes E)$

$$\sum_{k \geq 0} m_k(b^{\otimes k}) = 0 \qquad \bar{\partial}\varphi + [\varphi, \varphi] = 0$$

weakly unobs. $\quad \sum_{k \geq 0} m_k(b^{\otimes k}) = W(b)\, e$ $\quad$ projectively holom. VB $\quad F^{0,2} = \eta^{0,2} I_E$

$\rightsquigarrow \quad W : \mathcal{MC}_{weak}(L) \longrightarrow \mathbb{C} \quad (\text{or } \Lambda_0)$

Gluing $\mathcal{MC}_{weak}$'s (via wall crossing formula)

$\rightsquigarrow \quad W : \check{Y} \longrightarrow \mathbb{C}$

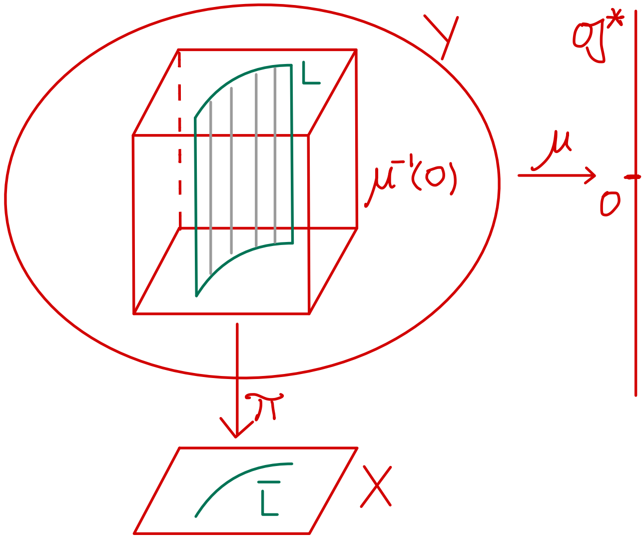
$$G \curvearrowright (Y, \omega) \xrightarrow{\mu} \mathfrak{g}^*$$ (assume $G \overset{\text{free}}{\curvearrowright} \mu^{-1}(0)$)

$$X = Y /\!\!/_0 G = \mu^{-1}(0)/G$$ sympl. quotient

If L is G-inv. & $L \subset \mu^{-1}(0)$

$$\downarrow \pi$$

then $L = \pi^{-1}(\bar{L}) \ \exists\ \bar{L} \overset{\text{Lagr.}}{\hookrightarrow} X$



$$\mathcal{MC}^G_{\text{weak}}(L)_{\mu=0} \xrightarrow{\cong} \mathcal{MC}_{\text{weak}}(\bar{L})$$

$$\mathcal{MC}^{G}_{\text{weak}}(L)_{\mu=0} \xrightarrow{\ \cong\ } \mathcal{MC}_{\text{weak}}(\bar{L})$$

$$\begin{tikzcd} L \arrow[r, hook, "G\text{-inv.}"] & \mu^{-1}(0) \arrow[r, hook] \arrow[d, "\pi"] & Y \\ \bar{L} \arrow[r, hook, "\text{Lagr.}"] & X & \end{tikzcd}$$

(i) $\mu = 0$ can be prescribed by "equivar." W,

(ii) $W^G_Y = W_X +$ correction terms

(vanish if X Fano)

$T \underset{\text{assume}}{=\!=} G \curvearrowright (Y, \omega) \xrightarrow{\mu} \mathfrak{g}^* \qquad \check{Y} := \{(L, \nabla)\} / \cong$

$L \overset{\text{Lagr.}}{\hookrightarrow} Y$ is T-inv.

$\Longrightarrow$ (i) $\mu|_L$ is a constant

(ii) deformations of L is also T-inv.

$\rightsquigarrow \quad F : \check{Y} \longrightarrow \check{T}_{\mathbb{C}} \cong \mathfrak{t}^* \times T$

$F(L, \nabla) = e^{\mu(L)} \cdot \mathrm{Hol}_T(\nabla)$

(L, ∇) : T-inv. Lagr. cycle w/ $L \subset \mu^{-1}(0)$

$\Longleftrightarrow (L, \nabla) \in F^{-1}(1)$

To determine $W_Y^G = W_X$ + correction terms we need cyclic element in an equivar. Lagrangian tri-module.

Lagrangian corresp

$$L_{12} \overset{\text{Lagr.}}{\subset} Y_1^- \times Y_2 \qquad \left(\text{write } Y_1 \xrightarrow{L_{12}} Y_2\right)$$

$$\Phi_{L_{12}} : \mathcal{F}uk(Y_1) \xrightarrow{?} \mathcal{F}uk(Y_2)$$

$$\Phi_{L_{12}}(L_1) \text{``}=\text{''} \pi_2(\pi_1^*(L_1) \cap L_{12})$$

need transversality condition

<u>Composition</u> of Lagr. corresp. $Y_1 \xrightarrow{L_{12}} Y_2$ and $Y_2 \xrightarrow{L_{23}} Y_3$

should be $Y_1 \xrightarrow{L_{13}} Y_3$ with L_{13} "$=$" $\pi_{13}\left(L_{12} \underset{Y_2}{\times} L_{23} \right)$

Transversality conditions :

$L_{12} \times L_{23}$ and $Y_1 \times \Delta_{Y_2} \times Y_3$ in $Y_1 \times Y_2 \times Y_2 \times Y_3$

transverse $\geqslant$ clean

§ Lagrangian correspondences for G-actions

$$G \curvearrowright (Y, \omega) \xrightarrow{\mu} \mathfrak{g}^*$$

$$X = \mu^{-1}(0)/G = Y /\!\!/_0 G$$

$\mu^{-1}(0) \subset Y$ coisotropic

But, $L^\pi := \mu^{-1}(0) \subset Y^- \times X$ Lagrangian

Give $Y \xrightarrow{L^\pi} X$

Given Lagrangian $L \subset Y$

$$G \curvearrowright Y \xrightarrow{\mu} \mathfrak{g}^*$$

$$X = \mu^{-1}(0)/G$$

If L is G-inv. & $L \subset \mu^{-1}(0)$

$$\mu^{-1}(0) \xrightarrow{\pi} X$$

then $L = \pi^{-1}(\bar{L})$ $\exists$ $\bar{L} \overset{\text{Lagr.}}{\subset} X$

$\bar{L} = L^{\pi} \circ L$ is a <u>clean</u> composition

$$\begin{tikzcd} pt \arrow[r, "L"] \arrow[rr, bend right, "\bar{L}"'] & Y \arrow[r, "L^{\pi}"] & X \end{tikzcd}$$

Equivar. Lagrangian Correspondence Tri-module

Theorem (Lau-L.-Li) $G_i \curvearrowright (Y_i, \omega_i) \xrightarrow{\mu_i} \mathfrak{g}_i^*$

$L_{ij} \subset Y_i \times Y_j$, $G_i \times G_j$-equivar. Lagr.

$$\begin{tikzcd} Y_1 \arrow[r, "L_{12}"] \arrow[rr, bend right, "L_{13} = L_{23} \circ L_{12}"'] & Y_2 \arrow[r, "L_{23}"] & Y_3 \end{tikzcd}$$

clean composition

$\Longrightarrow 1^\circ \ \exists \left(CF^{\bullet}_{equiv}(L_{13}; L_{12}, L_{23}), \{ n^{equiv}_{k''k'k} \} \right)$

A_∞-tri-module wrt $CF^{\bullet}_{equiv}(L_{13})_L \times CF^{\bullet}_{equiv}(L_{12})_R \times CF^{\bullet}_{equiv}(L_{23})_R$

(2° $\exists$ equivar. 'cyclic element' $\mathbb{1} \in CF^{\bullet}_{eq}(L_{13}; L_{12}, L_{23})$)

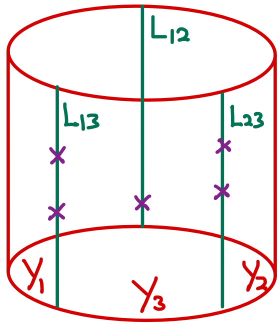
$$CF^{\bullet}_{equiv}(L_{13};L_{12},L_{23}) = \Omega^{\bullet}_{dR}\big((L_{23}\times L_{12}\times L_{13})_{equiv} \cap \Delta_{Y_1Y_2Y_3\,equiv}\big)$$

(inside $Y_1 Y_1 Y_2 Y_2 Y_3 Y_3$)

$$\eta^{equiv}_{k''k'k} : CF^{\bullet}_{eq}(L_{13})^{\otimes k''} \otimes CF^{\bullet}_{eq}(L_{13};L_{12},L_{23}) \otimes CF^{\bullet}_{eq}(L_{12})^{\otimes k'} \otimes CF^{\bullet}_{eq}(L_{23})^{\otimes k}$$

$$\longrightarrow CF^{\bullet}_{eq}(L_{13};L_{12},L_{23})$$

count holomorphic cylinders : out



$\mathcal{M}_{k'',k',k}(L_{13},L_{12},L_{23})$ in

Theorem (Lau - L. - Li) ... Continue

$$G_i \curvearrowright (Y_i, \omega_i) \xrightarrow{\mu_i} \mathfrak{g}_i^*$$

$$L_{ij} \subset Y_i \times Y_j$$

$$\begin{tikzcd} Y_1 \arrow[r, "L_{12}"] \arrow[rr, bend right, "L_{13} = L_{23} \circ L_{12}"'] & Y_2 \arrow[r, "L_{23}"] & Y_3 \end{tikzcd}$$

$\Longrightarrow$ 1° $\exists \left(CF^{\bullet}_{equiv}(L_{13} ; L_{12}, L_{23}), \{ n^{equiv}_{k'' k' k} \} \right)$

2° $\exists$ equivar. 'cyclic element' $\mathbb{1} \in CF^{\bullet}_{eq}(L_{13} ; L_{12}, L_{23})$

i.e. $n^{equiv}_{0,0,0}(\mathbb{1}) \equiv 0 \pmod{\Lambda_+}$

and $n^{equiv}_{1,0,0}(-, \mathbb{1}) : CF^{\bullet}_{eq}(L_{13}) \xrightarrow{\sim} CF^{\bullet}_{eq}(L_{13} ; L_{12}, L_{23})$

$$CF^1_G(L^\pi) \times CF^1_G(L) \xrightarrow{\ \circ\ } CF^1(\bar{L})$$

induces compositions of Maurer-Cartan elt.

$$\mathcal{MC}^G_{weak}(L^\pi) \times \mathcal{MC}^G_{weak}(L) \xrightarrow{\ \circ\ } \mathcal{MC}_{weak}(\bar{L}) \qquad (\text{up to } \cong)$$

$$W^G_{L^\pi}(b_{L^\pi}) + W^G_L(b_L) = W_{\bar{L}}(b_{L^\pi} \circ b_L)$$

via $O_{L^\pi_G}$, $\qquad \mathcal{MC}^G_{weak}(L) \longrightarrow \mathcal{MC}_{weak}(\bar{L})$

Ingredients of the proof:

1° For (i) $\mathcal{MC}(L) \subset \mathcal{MC}_{weak}(L)$

(ii) transverse ($\geqslant$ clean) composition

(iii) w/o group action,

this is a result of Fukaya.

2° Equivariant Floer theory

developed by Kim-Lau-Zheng

$$L \times_G E_G[N] \overset{\text{Lagr.}}{\subset} Y \times_G T^*(E_G[N])$$

finite dim. approx.

Question: Floer theory for $Y/\!\!/G$?

(Will assume $G = T$, say S^1)

Teleman conjecture. $Y \xleftrightarrow{MS} Y^\vee \xrightarrow{W} \mathbb{C}$

$S^1 \curvearrowright (Y, \omega) \xrightarrow{\mu} \mathbb{R} \ni \lambda$ regular value

$$\rightsquigarrow \begin{tikzcd} Y^\vee \arrow[d, "F"'] & \supset & Y^\vee_\lambda \arrow[d] \\ \mathbb{C}^\times & \ni & e^\lambda \end{tikzcd}$$

s.t.

$$(Y/\!\!/_\lambda S^1, \omega_{red}) \xleftrightarrow{MS} (Y^\vee_\lambda, W|_{Y^\vee_\lambda})$$

Eg. Hori-Vafa mirror for toric Fano.

Pomerleano-Teleman claimed closed string version

$QH^\bullet(Y/\!\!/_\lambda S^1)$ in Fano cases.

Remark: Mak-Pomerleano, ...

3d MS perspectives:

3d $\mathcal{N}=4$ A-twist pure S^1-gauge theory $\xleftrightarrow[A \qquad B]{\text{3d MS}}$ Rozansky-Witten 3d σ-model on $T^*\mathbb{C}^\times$

boundary conditions:

$$\begin{array}{c} S^1 \curvearrowright (Y,\omega) \\ \downarrow \mu \\ \mathbb{R} \end{array} \quad \overset{?}{\rightsquigarrow} \quad \begin{array}{c} Z \\ \downarrow \\ C \underset{\text{cpx. Lagr.}}{\subset} T^*\mathbb{C}^\times \end{array}$$

Eg. $S^1 \curvearrowright \{*\} \xrightarrow{\mu} \{\lambda\} \in \mathbb{R}$ $\qquad$ $\begin{array}{c} Z \\ \downarrow \| \\ T^*_{e^\lambda}\mathbb{C}^\times \subset T^*\mathbb{C}^\times \end{array}$

$$\begin{array}{c} Y^\vee = \{(L,\nabla)\} \xrightarrow{W} \mathbb{C} \\ \downarrow F \\ \mathbb{C}^\times \end{array} \longmapsto$$

Lagr. corresp.

$$\begin{array}{ccccc} & & T^*\check{Y} \times T^*\mathbb{C}^\times & & \\ & \swarrow & \cup \text{Lagr} & \searrow & \\ \mathrm{Gr}(dW) \subset T^*\check{Y} & & \mathcal{N}^*_{\mathrm{Gr}(F)/\check{Y}\times\mathbb{C}^\times} & & T^*\mathbb{C}^\times \supset C \end{array}$$

$\begin{array}{c} Z \\ \downarrow \\ C \end{array}$ $\leftarrow$ equal for transverse intersection

3d MS conjecture:

$$\mathrm{Hom}^{A}_{\text{gauge}}(\mathcal{B}_1, \mathcal{B}_2) = \mathrm{Hom}^{B}_{RW}(\mathcal{B}_1^\vee, \mathcal{B}_2^\vee)$$

intersect^{n} of cpx. Lagr.

In particular,

$$\underbrace{\mathrm{Hom}^{A}_{\text{gauge}}(S^1 \curvearrowright Y, \{\lambda\})}_{\text{Cat for: } \; Y/\!\!/_{\lambda} S^1} = \underbrace{\mathrm{Hom}^{B}_{RW}\left(\begin{matrix} Y^\vee \\ \downarrow \\ \mathbb{C}^\times \end{matrix}, W, T^*_{e^\lambda}\mathbb{C}^\times\right)}_{(Y^\vee_\lambda,\; W|_{Y^\vee_\lambda})}$$

Happy birthday to
Yau