Back to $\mathbb{F}_q$ - structures.

Let X be an affine variety endowed with an $\mathbb{F}_q$ - structure, i.e. with a Frob. endo. $F : X \to X$ over $\mathbb{F}_q$.

$$\mathcal{O}(X) = \mathbb{F} \otimes_{\mathbb{F}_q} A_q$$

$$A_q = \{ f \in \mathcal{O}(X) \mid \underset{f \circ F}{\underset{\|}{F^*(f)}} = f^q \}$$

Pick $X_1, \dots, X_n$ generators of A_q and $f_1, \dots, f_r \in \mathbb{F}_q[X_1, \dots, X_n]$ be generators of the ideal of relations.

$$X \hookrightarrow \mathbb{A}^n(\mathbb{F})$$

$$X = \{ (x_1, \dots, x_n) \in \mathbb{A}^n(\mathbb{F}) \mid f_1(x_1, \dots, x_n) = \dots = f_r(x_1, \dots, x_n) = 0 \}$$

$$F^*(X_i) = X_i^q \Rightarrow$$

$$F : X \longrightarrow X$$

$$(x_1, \dots, x_n) \longmapsto (x_1^q, \dots, x_n^q)$$

Theorem 3.5 (e). Let $F : X \to X$ be a Frobenius endomorphism over $\mathbb{F}_q$. Then F is bijective, but is not an isomorphism of varieties if $\dim X \geq 1$.

$$\underset{\underset{0}{\|}}{d_x F} : T_x X \longrightarrow T_{F(x)} X$$

/ tangent spaces

$$Z(X,t) = \exp\left(\sum_{n\geq 1} \frac{\# X(\mathbb{F}_{q^n})}{n} t^n\right)$$

Weil conjectures (1949); Dwork (1959) - Grothendieck (60s) - Deligne (1974) theorem

Assume that X is smooth, irreducible, projective of dim. d. Then:

(a) Rationality: $Z(X,t) \in \mathbb{Q}(t)$

(b) Functional equation: $Z\left(X, \frac{1}{q^d t}\right) = \pm q^{\frac{dE}{2}} t^E Z(X,t)$ with $E = -\deg Z(X,t)$

(c) Riemann Hypothesis: $Z(X,t) = \dfrac{P_1(t)\, P_3(t) \cdots P_{2d-1}(t)}{P_0(t)\, P_2(t) \cdots P_{2d}(t)}$

with $P_0(t) = 1 - t$; $P_{2d}(t) = 1 - q^d t$

and $P_i(t) = \prod_{j=1}^{\beta_i} (1 - \alpha_{ij} t)$ with $|\alpha_{ij}| = q^{i/2}$

(d) Topological interpretation: If X is a "reasonable reduction modulo p" of a smooth irreducible projective complex variety $\mathfrak{X}$, then

$$\beta_i = \dim H^i(\mathfrak{X}, \mathbb{Q}).$$

Theorem 3.7 (Grothendieck). Assume that $\dim X = d$ and let $\mathcal{J}(X)$ denote the set of its irreducible components of dim. d.

(a) $\dim H^i_c(X) < \infty$

(b) $H^i_c(X) = 0$ if $i < 0$ or $i > 2d$

(c) $H^{2d}_c(X) \underset{K\Gamma\text{-mod}}{\simeq} K[\mathcal{J}(X)]$ (permutation module)

(d) If $\mathcal{U} \subset X$ is open, Γ-stable, F-stable and $Z = X \setminus \mathcal{U}$, we have a long exact sequence

$$\cdots \to H^i_c(\mathcal{U}) \to H^i_c(X) \to H^i_c(Z) \to H^{i+1}_c(\mathcal{U}) \to \cdots$$

(e) Künneth formula:

$$H^n_c(X \times X') = \bigoplus_{i=0}^{n} H^i_c(X) \otimes H^{n-i}_c(X')$$

(f) Poincaré duality: if X is smooth, irreducible, projective, then there is a Γ-equivariant, F-equivariant duality

$$H^i_c(X) \times H^{2d-i}_c(X) \longrightarrow H^{2d}_c(X) \simeq K$$

(g) $H^i_c(\mathbb{A}^d(\mathbb{F})) = \begin{cases} K & \text{if } i = 2d \\ 0 & \text{if } i \neq 2d. \end{cases}$

(h) If X is smooth, irreducible, affine, then $H^i_c(X) = 0$ if $i < d$.

(i) $H^i_c(X)^\Gamma = H^i_c(X/\Gamma)$

(j) If Γ is contained in a connected algebraic group acting regularly on X, then Γ acts trivially on $H^i_c(X)$

Example 3.8.

$$\dim H^i_c(\mathbb{P}^1(\mathbb{F})) = \begin{cases} 1 & \text{if } i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Example 3.9. $X = \mathbb{P}^1(\mathbb{F}) \setminus \mathbb{P}^1(\mathbb{F}_q)$

$$\begin{cases} H^0_c(X) = 0 & \text{(smooth affine)} \\ H^1_c(X) \simeq \mathrm{St} & \text{as a } KG\text{-module} \\ H^2_c(X) \simeq K_G & \text{as a } KG\text{-module} \end{cases}$$

$$K = \overline{\mathbb{Q}}_\ell$$

For $f \in \Gamma$ or f a Frob. endo. $/\mathbb{F}_q$, let

Lefschetz number ↙ $\mathrm{Tr}^*_X(f) = \sum_{i \geq 0} (-1)^i \, \mathrm{Tr}(f, H^i_c(X))$

$(\mathrm{Tr}^*_X(\mathrm{Id}_X) = \text{Euler characteristic})$

Theorem 3.10.

(a) If $\mathcal{U}$ is an f-stable open subset and $Z = X \setminus \mathcal{U}$, then

$$\mathrm{Tr}^*_X(f) = \mathrm{Tr}^*_{\mathcal{U}}(f) + \mathrm{Tr}^*_Z(f)$$

(b) If s and u are two commuting auto. of X such that $\mathrm{Gcd}(p, o(s)) = 1$ and $o(u) = p^?$, then

$$\mathrm{Tr}^*_X(su) = \mathrm{Tr}^*_{X^s}(u)$$

(c) If S is a torus (i.e. $\simeq (\mathbb{F}^\times)^r$ for some r) acting on X and commuting with some $\gamma \in \Gamma$, then

$$\mathrm{Tr}^*_X(\gamma) = \mathrm{Tr}^*_{X^S}(\gamma).$$

(d) If $\gamma \in \Gamma$, then $\mathrm{Tr}^*_X(\gamma) \in \mathbb{Z}$.

In particular, $\mathrm{Tr}^*_X(\gamma) = \mathrm{Tr}^*_X(\gamma^{-1})$

Theorem 3.11. (a) $\mathrm{Tr}^*_X(F) = |X^F|$

(b) If X is irreducible of dim. d, then F acts on $H^{2d}_c(X) \simeq \overline{\mathbb{Q}}_\ell$ by mult. by q^d.

(c) The eigenvalues of F on $H^i_c(X)$ are of the form $\omega q^{j/2}$, where $0 \leq j \leq i$ and ω is an algebraic number all of whose Galois conjugate have norm 1.

If X is smooth, irreducible and projective, then $j = i$.

(d) If X is the "reduction mod p" of a smooth projective complex algebraic variety, then $H^i_c(X) \simeq K \otimes_{\mathbb{Z}} H^i_c(X, \mathbb{Z})$

(a), (b) : Grothendieck
(c) : Deligne
(d) : Grothendieck

Corollary 3.12. The Frobenius endo. F induces an automorphism of $H^i_c(X)$.

Proof of Weil conjectures.

Let $\beta_i = \dim H^i_c(X)$

Let $(\alpha_{ij})_{1 \le j \le \beta_i}$ denote the eigenvalues of F (with multiplicities)

Then, by 3.11(a),

$$\# X(\mathbb{F}_{q^n}) = \mathrm{Tr}^*_X(F^n) = \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{\beta_i} \alpha_{ij}^n$$

$$\Rightarrow \quad Z(X,t) = \prod_{i=0}^{2d} \left(\prod_{j=1}^{\beta_i} (1 - \alpha_{ij} t) \right)^{(-1)^{i+1}}$$

$\Rightarrow \quad Z(X,t) \in K(t) \cap \mathbb{Q}[[t]] = \mathbb{Q}(t)$.

$\Rightarrow$ (a).

Now, by Poincaré duality (3.7(f)): $H^i_c(X) \times H^{2d-i}_c(X) \longrightarrow H^{2d}_c(X) \simeq K$

F acts by mult. by q^d (3.11(b)).

$\Rightarrow \quad \beta_i = \beta_{2d-i}$ and $\alpha_{ij} = \dfrac{q^d}{\alpha_{2d-i,j}}$

exercise $\Rightarrow$ Functional equation (b).

3.11(c) implies Weil conjecture (c)

(d) — — — — — (d). ■

Theorem 3.11. (a) $\mathrm{Tr}^*_X(F) = |X^F|$

(b) If X is irreducible of dim. d, then F acts on $H^{2d}_c(X) \simeq \overline{\mathbb{Q}}_\ell$ by mult. by q^d.

(c) The eigenvalues of F on $H^i_c(X)$ are of the form $\omega q^{j/2}$, where $0 \le j \le i$ and ω is an algebraic number all of whose Galois conjugate have norm 1.
If X is smooth, irreducible and projective, then $j = i$.

(d) If X is the "reduction mod p" of a smooth projective complex algebraic variety, then $H^i_c(X) \simeq K \otimes_{\mathbb{Z}} H^i_c(X, \mathbb{Z})$

Remarks 3.13. (1) If X is smooth projective, then Weil conjectures imply that β_i depend only on $\left(\# X(\mathbb{F}_{q^n})\right)_{n\geq 1}$.

So β_i does not depend on the choice of ℓ.

(2) Tate conjecture (1963) $\Rightarrow$ F is diagonalizable on $H^i_c(X)$.

STILL UNSOLVED !!

(3) An algebraic number $\omega \in \overline{\mathbb{Q}}$ such that $|\sigma(\omega)| = 1$, $\forall \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is not necessarily a root of unity (example: $\frac{1}{5}(3+4i)$)

Exercise. An alg. integer satisfying the same property is a root of unity.

(4) If X is smooth, projective, irreducible, then $\dim H^0_c(X) = 1$ and F acts on $H^0_c(X)$ by multiplication by 1.

(5) If X is irreducible, then $\dfrac{\# X(\mathbb{F}_{q^n})}{q^{dn}} \xrightarrow[n\to\infty]{} 1$ where $d = \dim X$.

(6) If $\mathcal{C}$ is a smooth projective curve of genus g, then 3.11(d) $\Rightarrow$ $\dim H^1_c(\mathcal{C}) = 2g$, so

$$\# \mathcal{C}(\mathbb{F}_q) = Tr(F, H^0_c(\mathcal{C})) - Tr(F, H^1_c(\mathcal{C})) + Tr(F, H^2_c(X)) = 1 + q - (\alpha_{11} + \alpha_{12} + \dots + \alpha_{1,2g})$$

$\Rightarrow$ HASSE - WEIL THEOREM

Theorem 3.11. (a) $Tr^*_X(F) = |X^F|$

(b) If X is irreducible of dim. d, then F acts on $H^{2d}_c(X) \simeq \overline{\mathbb{Q}}_\ell$ by mult. by q^d.

(c) The eigenvalues of F on $H^i_c(X)$ are of the form $\omega\, q^{j/2}$, where $0 \leq j \leq i$ and ω is an algebraic number all of whose Galois conjugate have norm 1.
If X is smooth, irreducible and projective, then $j = i$.

(d) If X is the "reduction mod p" of a smooth projective complex algebraic variety, then $H^i_c(X) \simeq K \otimes_{\mathbb{Z}} H^i_c(X, \mathbb{Z})$

PART I. $SL_2(\mathbb{F}_q)$

Chapter 1. Structure

Chapter 2. Harish-Chandra induction

Chapter 3. Introduction to ℓ-adic cohomology

TODAY AND FRIDAY (Chapter 4. Deligne-Lusztig theory for $SL_2(\mathbb{F}_q)$

Chapter 4. Deligne-Lusztig induction

4.A. Drinfeld curve.

$G = SL_2(\mathbb{F}_q)$

$$Y = \{ (x,y) \in \mathbb{A}^2(\mathbb{F}) \mid xy^q - yx^q = 1 \}$$

(Drinfeld curve: 1974)

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{G} = GL_2(\mathbb{F}_q)$ acts on $\mathbb{A}^2(\mathbb{F})$

$$(ax+by)(cx+dy)^q - (cx+dy)(ax+by)^q$$
$$= (ax+by)(cx^q+dy^q) - (cx+dy)(ax^q+by^q)$$
$$= (ad-bc)(xy^q - yx^q)$$

So G acts on Y

Also μ_{q+1} acts also on Y by homothety.
($(q+1)$-th roots of unity)

Both actions commute!

We get an action of $G \times \mu_{q+1}$ on Y.
(G acting on the left; μ_{q+1} acting on the right)

Proposition 4.1. (a) The curve Y is smooth, affine, irreducible.

(b) G acts freely on Y (i.e. $\mathrm{Stab}_G(a) = 1$ for all $a \in Y$)

(c) μ_{q+1} acts freely on Y

<u>Remark</u> (exercise). The group $(G \times \mu_{q+1}) / \langle (-I_2, -1) \rangle$ does not act freely.

Proof. (a) Y is affine by definition.

Let $f(x,y) = xy^q - yx^q - 1$ and let P be a non-constant divisor of f.

Since $P(0,0) \neq 0$ so we can assume that $P(0,0) = 1$. Then:

- $P(x,0)$ divides $f(x,0) = -1 \Rightarrow P(x,0) = 1$

$\Rightarrow$ y divides $P(x,y) - 1$.

- If $a \in \mathbb{F}_q$, $P(ay,y)$ divides $f(ay,y) = -1$ so $P(ay,y) = 1$.

$\Rightarrow$ $x - ay$ divides $P(x,y) - 1$

So $y \prod_{a \in \mathbb{F}_q} (x - ay)$ divides $P(x,y) - 1$.

So $\deg P \geqslant q+1$: f is irreducible.

$$Y_{sing} = \{ (x,y) \in \mathbb{A}^2(\mathbb{F}) \mid f(x,y) = \underset{\substack{\| \\ y^q}}{\frac{\partial f}{\partial x}(x,y)} = \underset{\substack{\| \\ -x^q}}{\frac{\partial f}{\partial y}(x,y)} = 0 \} = \emptyset.$$

(b) Let $g \in G$ be such that $Y^g \neq \emptyset$. We need to prove that $g = I_2$.

Since $(0,0) \notin Y$ and $Y^g \neq \emptyset$

$\Rightarrow$ g admits 1 as an eigenvalue.

But $\det(g) = 1$ so the "other" eigenvalue is also 1.

So $g \sim u_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $u_- = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix}$ or I_2.

But $Y^{u_+} = \{ (x,y) \in Y \mid y = 0 \} = \emptyset$

and $Y^{u_-} = \emptyset$.

So $g = I_2$.

(c) is easy. ∎

Remark 4.2. It is defined over $\mathbb{F}_q$ (!!)

$$F : Y \longrightarrow Y$$
$$(x,y) \longmapsto (x^q, y^q)$$

"natural" $\mathbb{F}_q$-structure.

On Y, $F \circ (g, \xi) = (g, \xi^q) \circ F$
$= (g, \xi^{-1}) \circ F$

4. B. Definition.

$H^i_c(Y)$ is $(KG, K\mu_{q+1})$ - bimodule

$$\mathcal{R}'_i = \mathcal{F}_{H^i_c(Y)} : K\mu_{q+1}\text{-mod} \longrightarrow KG\text{-mod}$$

$$K\, T'^{12}$$

$$R'_i : \mathrm{Class}(T') \longrightarrow \mathrm{Class}(G)$$

Definition. The Deligne-Lusztig induction is the map

$$R' : \mathrm{Class}(T') \longrightarrow \mathrm{Class}(G)$$

$$\| $$

$$\sum_{i \geqslant 0} (-1)^{i+1} R'_i = R'_1 - R'_2 .$$

$$R'(\theta)(g) = -\frac{1}{q+1} \sum_{t \in T'} \mathrm{Tr}^*_Y(g, t)\ \theta(t^{-1})$$

$$\begin{cases} R : \mathrm{Class}(T) \longrightarrow \mathrm{Class}(G) \\ R(\mathrm{Irr}\, T) \subset \mathbb{N}\, \mathrm{Irr}(G) \end{cases}$$

$$R'(\mathrm{Irr}\, T') \subset \mathbb{Z}\, \mathrm{Irr}(G)$$

$$\langle R(\alpha), R'(\theta)\rangle_G = 0$$

Harish-Chandra induction.

$\mathcal{R} : KT\text{-mod} \longrightarrow KG\text{-mod}$

$\quad\quad\; \| \quad K\mu_{q-1}\text{-mod}$

$R : \mathrm{Class}(T) \longrightarrow \mathrm{Class}(G)$

<u>Prop. 2.4.</u> $R(\alpha)(1) = (q+1)\,\alpha(1)$

<u>Mackey formula</u> 2.5. $\langle R(\alpha), R(\beta)\rangle_G = \langle \alpha, \beta\rangle_T + \langle \alpha, {}^s\beta\rangle_T$

<u>Theo. 2.6.</u> Let $\alpha, \beta \in \mathrm{Irr}(T) = \mathrm{Hom}(T, K^\times)$.

(a) $R(\alpha) = R(\alpha^{-1})$

(b) $\langle R(\alpha), R(\beta)\rangle = 0$ if $\beta \neq \alpha^{\pm 1}$

(c) if $\alpha^2 \neq 1$, then $R(\alpha) \in \mathrm{Irr}(G)$

<u>Example 2.7.</u> $R(1_T) = 1_G + St$

<u>Example 2.8.</u> $R(\alpha_0) = R(\alpha_0)^+ + R(\alpha_0)^-$

(2.9) $R(\chi_{KT}) = \underset{\text{dim. }1}{1_G} + \underset{\text{dim. }q}{St} + \underset{\text{dim. }\frac{q+1}{2}}{R(\alpha_0)^+} + \underset{\text{dim. }\frac{q+1}{2}}{R(\alpha_0)^-}$

$$+\, 2 \sum_{\alpha \in (\mathrm{Irr}\, T \setminus \{1, \alpha_0\})/\mathrm{inv}} \underset{\text{dim } q+1}{R(\alpha)}$$

$\frac{q+5}{2}$ distinct irreducible characters.

Deligne-Lusztig induction

$\mathcal{R}'_i : KT'\text{-mod} \longrightarrow KG\text{-mod}$

$\quad\quad\; \| \quad K\mu_{q+1}\text{-mod}$

$R' = R'_1 - R'_2 : \mathrm{Class}(T') \longrightarrow \mathrm{Class}(G)$

$R'(\theta)(1) = (q-1)\,\theta(1)$

<u>Mackey formula.</u> $\langle R'(\theta), R'(\eta)\rangle_G = \langle \theta, \eta\rangle_{T'} + \langle \theta, {}^{s'}\eta\rangle_{T'}$

<u>Theo.</u> Let $\theta, \eta \in \mathrm{Irr}(T') = \mathrm{Hom}(T', K^\times)$

(a) $R'(\theta) = R'(\theta^{-1})$

(b) $\langle R'(\theta), R'(\eta)\rangle = 0$ if $\theta \neq \eta^{\pm 1}$

(c) If $\theta^2 \neq 1$, then $R'(\theta) \in \mathrm{Irr}\, G$

<u>Example.</u> $R'(1_{T'}) = -1_G + St$

<u>Example.</u> $R'(\theta_0) = R'(\theta_0)^+ + R'(\theta_0)^-$

$R'(\chi_{KT'}) = -\underset{\text{dim. }1}{1_G} + \underset{\text{dim. }q}{St} + \underset{\text{dim. }\frac{q-1}{2}}{R'(\theta_0)^+} + \underset{\text{dim }\frac{q-1}{2}}{R'(\theta_0)^-}$

$$+\, 2 \sum_{\theta \in (\mathrm{Irr}\, T \setminus \{1, \theta_0\})/\mathrm{inv}} \underset{\text{dim. } q-1}{R'(\theta)}$$

$\frac{q+3}{2}$ distinct NEW irreducible characters