Critical Phenomena and universality

- Experimental evidence for the critical exponent.
- Universality: Many systems have the same exponent.
- Relationship between critical exponents.

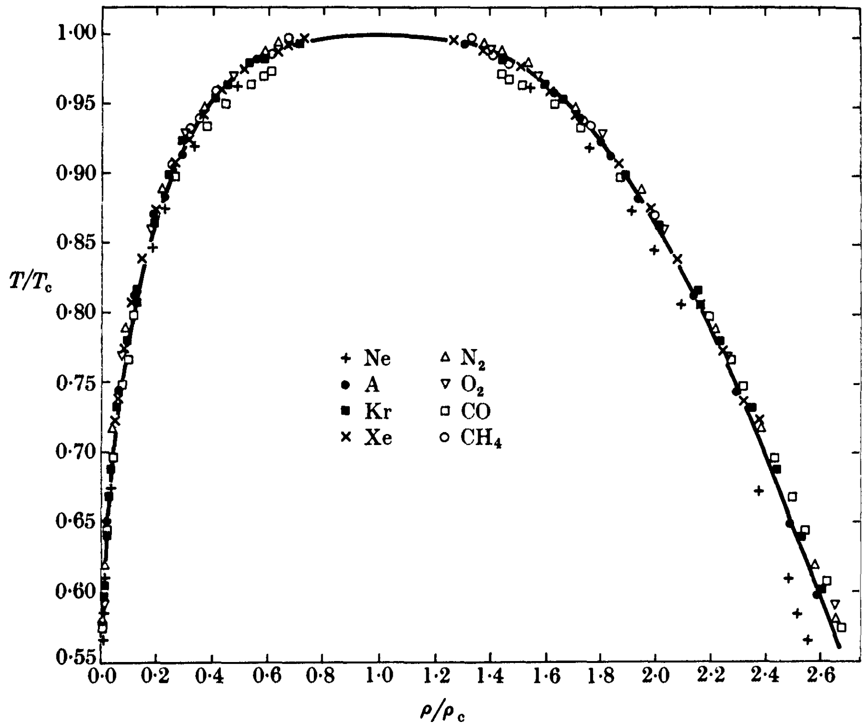


FIG. 1.8. Measurements on eight fluids of the coexistence curve (a reflection of the $P\rho T$ surface in the ρT plane analogous to Fig. 1.3). The solid curve corresponds to a fit to a cubic equation, i.e. to the choice $\beta = \frac{1}{3}$, where $\rho - \rho_c \sim (-\epsilon)^\beta$. From Guggenheim (1945).

is the zero-field magnetization M because M is a measure of the degree to which the magnetic moments are aligned throughout the crystal. Here again the classic Weiss theory predicts a quadratic dependence $M^2 \propto (T_c - T)$, whereas we see that $M^3 \propto (T_c - T)$ would seem to be an appropriate fit to the measurements of Heller and Benedek shown in Fig. 1.9.

It is customary to say that the order parameter varies as $(-\epsilon)^\beta$ where

$$\epsilon \equiv \frac{T - T_c}{T_c} \tag{1.2}$$

and where the critical-point exponent β typically has a value in the range 0·3–0·5. It is important to stress that it is not necessary to have a strict proportionality between the order parameter and $(-\epsilon)^\beta$ in order to be able to define a critical-point exponent. In fact, if we knew that a simple relation of the form $M = \mathscr{B}(-\epsilon)^\beta$ were valid, then three measurements in the critical region would suffice to determine the exponent β! In practice there are frequently correction terms, so that M might have the form $\mathscr{B}_0(-\epsilon)^\beta\{1 + B(-\epsilon)^x + \ldots\}$ with $x > 0$. Hence a more natural definition of the critical-point exponent β is

$$\beta \equiv \lim_{\epsilon \to 0} \frac{\ln M}{\ln(-\epsilon)}, \tag{1.3}$$

where the correction terms will drop out in taking the limit. In fact, critical-point exponents are frequently determined by measuring the slopes of log–log plots of experimental data, since l'Hospital's rule, together with eqn (1.3), implies that $\beta = d(\ln M)/d\{\ln(-\epsilon)\}$. Although this is a particularly quick method of determining the exponent, it requires a prior knowledge of the critical temperature, so that in practice one must frequently resort to plotting $M^{1/\beta}$ for several trial values of β until a value is found which produces a straight line.

At one time many workers believed that all materials have the same exponents. For example, we remarked above that all eight fluids shown in the Guggenheim plot, Fig. 1.8, appear to have roughly the same exponent, $\beta \simeq \frac{1}{3}$. Hence it was rather satisfying when the first accurate measurements of β for a magnetic system, those of Heller and Benedek in 1962, produced the value $\beta = 0{\cdot}335 \pm 0{\cdot}005$ (cf. Fig. 1.9), and

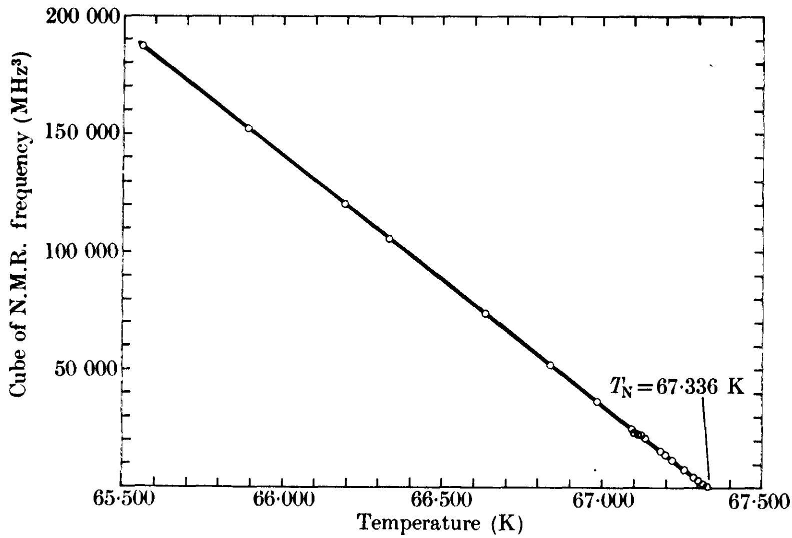


FIG. 1.9. Dependence upon temperature of the cube of the zero-field magnetization for MnF_2. Since MnF_2 is an antiferromagnet instead of a ferromagnet, the critical temperature is denoted by T_N rather than by T_c. After Heller and Benedek (1962).

subsequent measurements on other magnetic systems also appeared to yield similar values of β. However, this once-hoped-for universality has yet to be more rigorously demonstrated, and there now exists a growing list of materials for which $\beta = \frac{1}{3}$ is definitely outside the experimental error. For example, particularly accurate measurements supporting $\beta = 0{\cdot}354$ for helium are shown in Fig. 1.10.

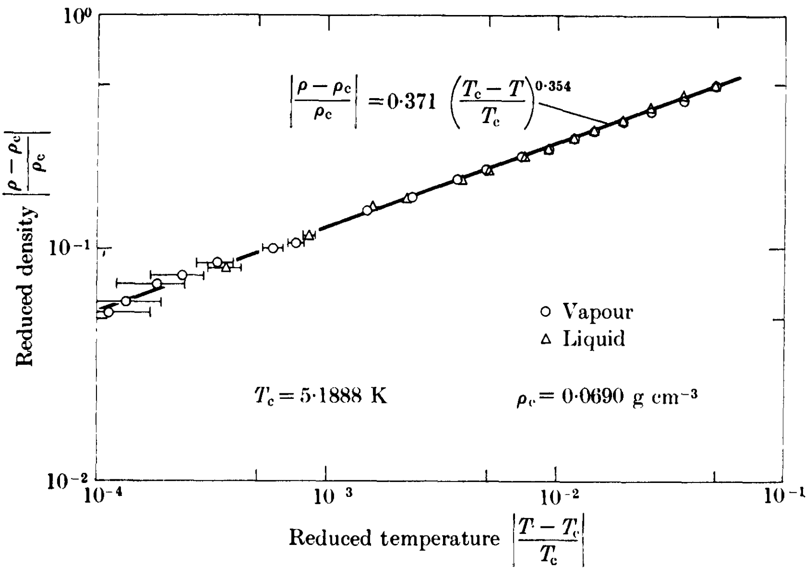


FIG. 1.10. Measurements of the coexistence curve of helium in the neighbourhood of its critical-point. The critical-point exponent β has a value of 0·354. After Roach (1968).

For some of these materials, however, the source of the discrepancy may be due to complicating factors such as the lattice compressibility. In Fig. 1.11, for example, are shown some recent data on the magnetization of CoO, which contracts suddenly on cooling through the critical temperature so that the exchange energy between neighbouring atomic moments increases. Hence when the critical temperature is approached from below, the system finds the exchange energy and hence the effective critical temperature decreasing (kT_c is generally thought to be a linear function of J, as one might imagine from dimensional analysis), and the measured critical-point exponent β is decreased below what one would expect for an incompressible lattice. Thus the value $\beta = 0{\cdot}244 \pm 0{\cdot}015$ is obtained from the slope of the log–log plot of the CoO data in Fig. 1.11 whereas, when corrected for this lattice contraction effect, the data indicate $\beta = 0{\cdot}290 \pm 0{\cdot}025$.

In Chapter 3 we shall define a good many of the commonly used critical-point exponents—suffice it to say here that there are essentially as many exponents as there are singular functions, and the Greek alphabet is fast being exhausted. Three of the most common critical-point exponents—α', β, and γ'—are defined for fluid and magnetic systems in Table 1.1. Note that minus signs are associated with the

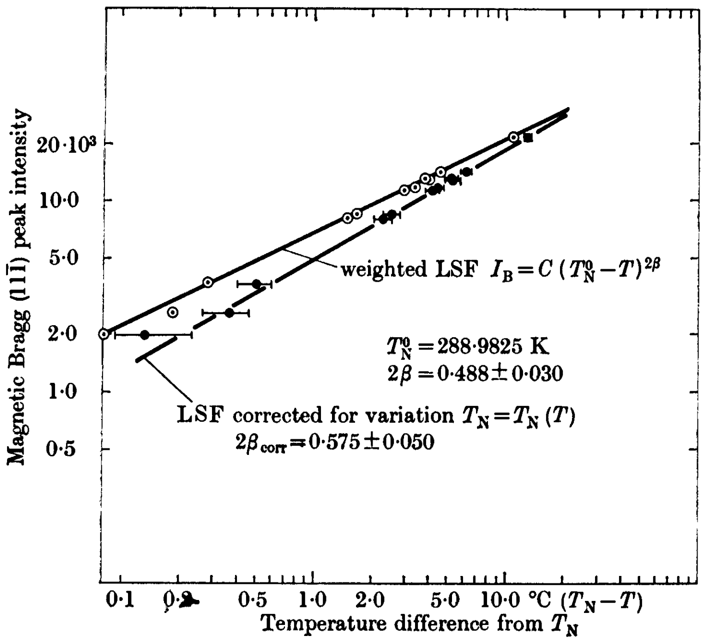


FIG. 1.11. Dependence of the logarithm of the magnetic Bragg peak intensity from neutron scattering from CoO upon the logarithm of ($T_N - T$). This intensity is proportional to the square of the spontaneous magnetization. The upper curve is a least squares fit (LSF) to a power law, assuming that the lattice is incompressible. The lower curve is a similar least squares fit to data that have been corrected for the lattice compressibility. The critical exponent obtained from the lower curve is clearly in better agreement than the upper curve with the anticipated value of $\beta \simeq 0{\cdot}3$. After Rechtin, Moss, and Averbach (1970).

exponents for response functions such as the specific heat, compressibility, and susceptibility which are expected, theoretically speaking, to diverge to infinity at the critical point; hence the exponents α' and γ' are defined such that they are positive quantities. Of course, no one has ever measured an infinite value for any of these response functions. This is not only because we never can make measurements arbitrarily close to T_c (measurements for $\epsilon < 10^{-6}$ or closer than one part in a

TABLE 1.1

Representative critical-point exponents for fluid and magnetic systems. For simplicity we have only considered here the approach to T_c from the low-temperature side. More complete tables are shown in Chapter 3

Definition	α'	β	γ'	$\alpha'+2\beta+\gamma'$
Fluid	$C_{V=V_c} \sim (-\epsilon)^{-\alpha'}$	$\rho_L - \rho_G \sim (-\epsilon)^{\beta}$	$K_T \sim (-\epsilon)^{-\gamma'}$	—
Magnet	$C_{H=0} \sim (-\epsilon)^{-\alpha'}$	$M_{H=0} \sim (-\epsilon)^{\beta}$	$\chi_T \sim (-\epsilon)^{-\gamma'}$	—
Typical experimental values				
Fluid or magnet	0–0·2	0·3–0·5	1·1–1·4	$\lesssim 2$
Theories				
van der Waals or Weiss	0 (discontinuity)	$\frac{1}{2}$	1	2
Two dimensional Ising model	0 (log)	$\frac{1}{8}$	$\frac{7}{4}$	2

million from the critical temperature are extremely rare) but also because some sort of rounding-off of the data is frequently found, as we see, for example, in the specific-heat data shown in Fig. 1.12.

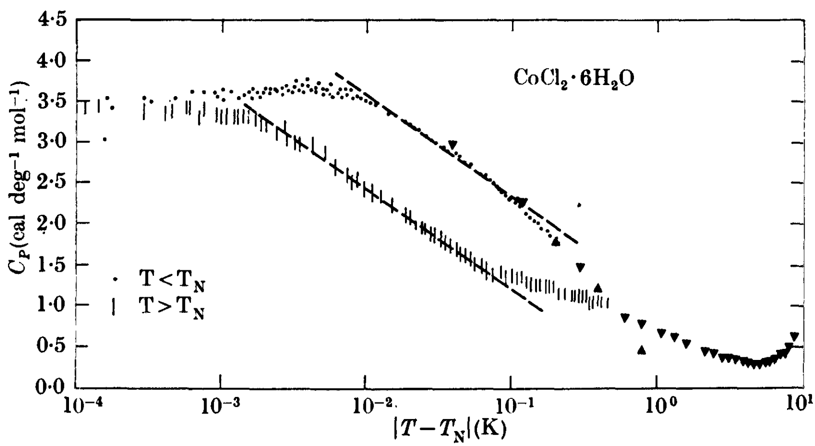


FIG. 1.12. Dependence of the specific heat upon the logarithm of $|T - T_N|$ for cobalt chloride. The data appear to be fitted fairly well by a logarithmic divergence except within a few millidegrees of T_N. After Kadanoff *et. al.* (1967).

1.2.2. *Results from model systems*

The number of model systems which have been studied as a means of gaining insight into the nature of phase transitions and critical phenomena is extremely large and therefore we shall limit our remarks here to the two models we introduced above, the Ising model and the Heisenberg model. Although both these models were proposed in the early years of this century, it is only within the last two decades that much of their richness has come to be appreciated.

The highlight in any discussion of the Ising model is perforce

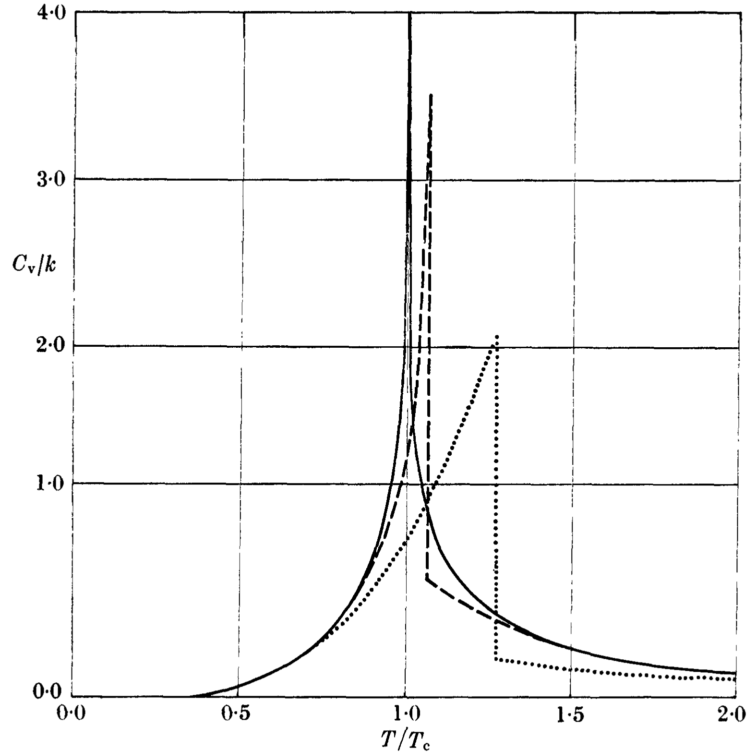


FIG. 1.13. The solid curve shows the specific heat of the two-dimensional Ising model as obtained from the exact solution of Onsager (solid curve), from the Bethe approximation (dotted curve), and the Kramers–Wannier and Kikuchi approximation (broken curve). After Domb (1960).

Onsager's solution, in 1944, for the $H = 0$ partition function of a two-dimensional lattice. From the partition function he was able to demonstrate that the specific heat possesses a logarithmic divergence at T_c when approached from either side of the transition. This result stood in dramatic contradistinction to the predictions of the mean field theory and other theories of cooperative phenomena of that day which predicted a simple discontinuity in the specific heat (cf. Fig. 1.13). In

Understanding critical phenomena

- Renormalization group
- An example: the Kosterlitz Thouless transition.

KT: Two dimensional melting

- In two dimension, no long range order
- What is the order parameter?
- Fluids have zero shear modulus, solids do not.
  Kosterlitz-Thouless, Part of their Nobel Prize

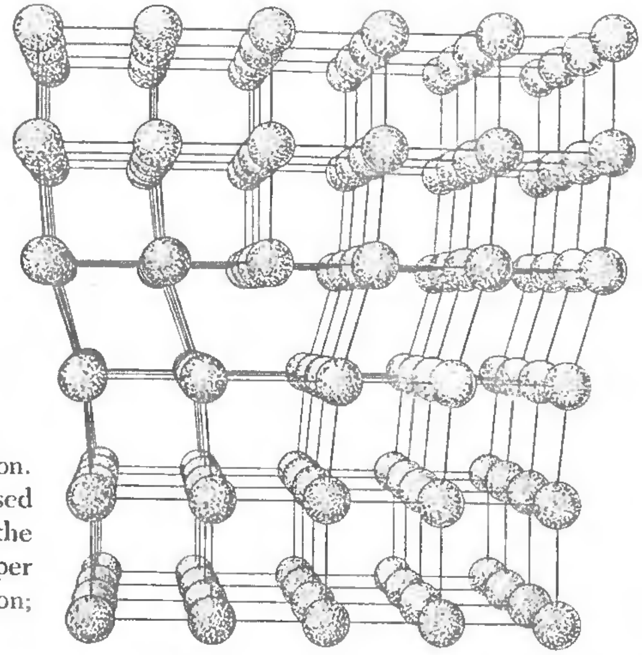


Figure 4 Structure of an edge dislocation. The deformation may be thought of as caused by inserting an extra plane of atoms on the upper half of the y axis. Atoms in the upper half-crystal are compressed by the insertion; those in the lower half are extended.

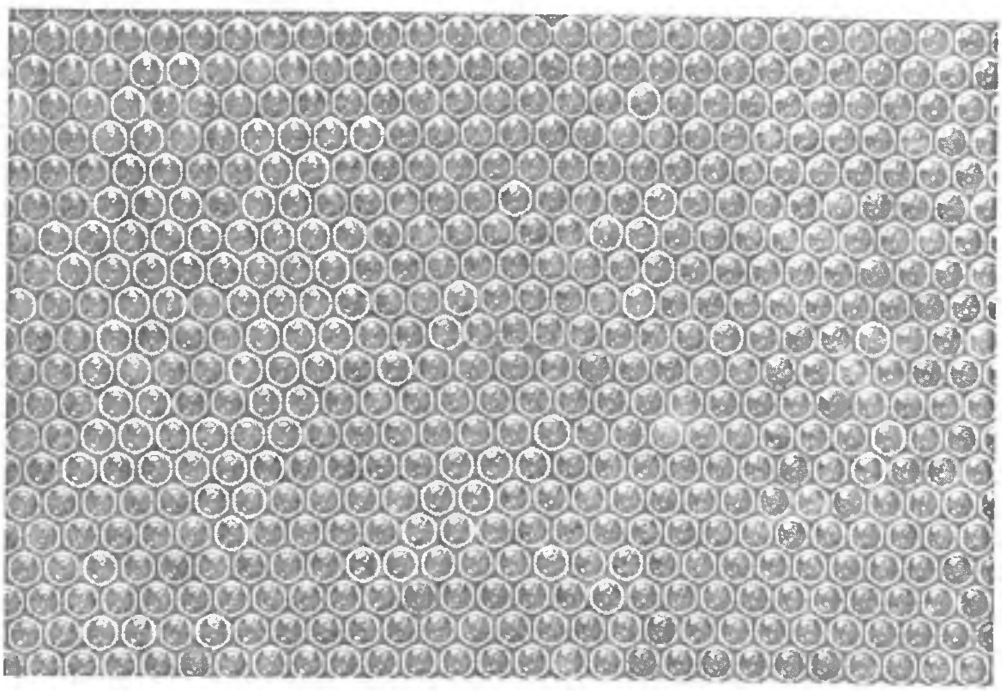


Figure 5 A dislocation in a two-dimensional bubble raft. The dislocation is most easily seen by turning the page by 30° in its plane and sighting at a low angle. (W. M. Lomer, after Bragg and Nye.)

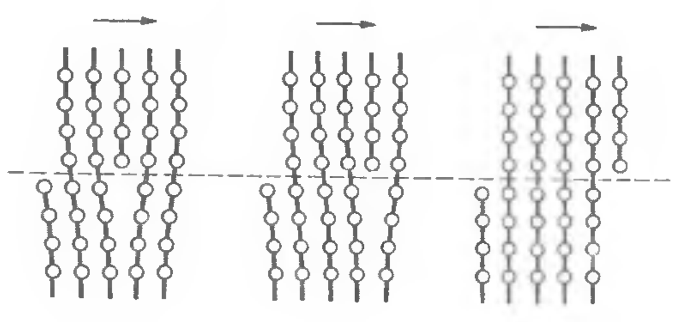


Figure 6 Motion of a dislocation under a shear tending to move the upper surface of the specimen to the right. (D. Hull.)

Understanding critical phenomena

- Renormalization group
- An example: the Kosterlitz Thouless transition.

KT: Two dimensional melting

- In two dimension, no long range order
- What is the order parameter?
- Fluids have zero shear modulus, solids do not

KT: Two dimensional melting

- In two dimension, no long range order
- What is the order parameter?
- Zero shear modulus, fluid
- Dislocation unbinding. Dislocations are topological defects. One has to remove a whole line of atoms to get back to the perfect solid and cannot get back to the perfect solid by continuous deformation of the system.

What is a renormaliation group (RG) transformation?

- The statistical averages such as the calculation of the partition function involves summing over all possible configurations for the variables of interest.
- We do a partial sum over configurations of a certain length scale in the (grand) partition function, usually with some approximations.
- We then redefine parameters so that the result of the partial sum involves an energy of the same functional form as the original energy.
- The relationship between the new and original parameters is called a RG transformation.

KT RG

- We consider a collection of dislocations so the energy of two dislocations is (for a pair of opposute charge, $\mathrm{q} = -\mathrm{q}'$)

$H_0 = 2q'q \quad \ln\left(\frac{r}{r_0}\right) + 2\mu'$, for r>$r_{0,}$ the size of a dislocation.

- Assume low density with , $y_0 = e^{-\beta\mu\prime}$ small, $\beta = 1/kT$
- Do RG transformation by integrating out in the grand partition function configurations with the pairs of opposite charges that are close by with r_0<r<r_0+dr. The length scale of the problem, r_0, is increased but the free energy is of the same form .
- We have renormalized pairs with renormalized r_0, y_0 and a renormalized coupling $\mathrm{K} = \beta q^2$, $\beta = \frac{1}{kT}$.

KT: screening by other pairs

- $H_0 = 2q'q\ln\left(\frac{r}{r_0}\right) + 2\mu$' for the dislocations of charges q and q'
- The effect of the close by pairs of ooposite charges is to screen out the the interaction the charges and make q smaller.
- The free energy depends on $e^{-\beta H}$. We perform the RG transformation on the screening term /kT so that the functional ln r form is maintained.
- Ln(r) is of the same form as the electric potential between charged rods. We shall us the language of electrostatics.

The screening is from the polarizability of pairs

- The polarizability p (change of the dipole moment per unit electric field) of a pair of charges $\pm q$ separated by a distance r but randomly oriented (diple moment $\vec{m} = q\vec{r}$) under an electric field **E** is determined as follows:
- Energy $\mathrm{H} = -\mathrm{Eqrcos\theta} + other\ terms$
- $\mathrm{p} = \mathrm{q}\partial_E < \mathrm{rcos\theta} >$
- $< \mathrm{rcos\theta} >= \int e^{-H/kT}\ \mathrm{rcos\theta} / \int e^{-H/kT} , \beta = 1/kT$
- $\partial_E < \mathrm{rcos\theta} > = q \int e^{-H/kT} (\ \mathrm{rcos\theta})^2 / \mathrm{kT} / \int e^{-H/kT}$
- $< \cos^2\theta > = 1/2$
- $\mathrm{p} = \mathrm{q}\partial_E < \mathrm{rcos\theta} > = (\ \mathrm{qr})^2 / 2\mathrm{kT}$

- $\mathrm{p} = \beta q^2 r^2/2$,
- Density of dislocation pairs in annulus of width dr:
- $dn = 2\pi r dr e^{-\beta H_0}/r_0^4$
- Susceptibility from pairs in an annulus of width dr:
- $d\chi = pdn = K\pi dx y_0^2 x^{-2K+3}$, $x = r/r_0$, $y_0 = e^{-\beta\mu\prime}$
- Coupling $\mathrm{K} = \beta q^2$ due to screening: $K^s = K - \int_1^\infty d\chi$
- $1/K^s = 1/(K - \int_1^\infty d\chi) = \frac{1}{K} + \int_1^\infty d\chi/\mathrm{K}$,

Do RG transformation with.

- $1/K^s = 1/K + \int_1^\infty d\chi/K$ (1)
- $d\chi/K = \pi dx y_0^2 x^{-2K+3}$
- Do RG transformation by integrating out contributions with x from 1 to b:
- $1/K^s = 1/K + \int_1^b d\chi/K + \int_b^\infty d\chi/K$
- $dK^{-1} = \int_1^b d\chi/K = (b-1)\pi y_0^2$
- Introduce renormalzized r_0, y_0 so the new equation looks the same as eq. (1).
- $\int_b^\infty d\chi/K = \int_1^\infty \pi du z_0^2 u^{-2K+3}$, $u = \frac{x}{b}$, $z_0^2 = y_0^2\ b^{-2K+4}$,
- $dy_0^2 = y_0^2\ (b^{-2K+4} - 1)$

RG eqn

- $dK^{-1} = (b-1)\pi y_0^2$
- $dy_0^2 = y_0^2 \,(b^{-2K+4} - 1)$
- Let b=1+dl,
- $dK^{-1}/dl = \pi y_0^2$
- $dy_0 \,/\mathrm{dl} = y_0 \,(2-K)$
- At K=2, y=0, the parameters do not change. This is called a fixed point of the transformation.

RG eqn

- $dK^{-1}/dl = \pi y_0^2$
- $dy_0\ /\mathrm{dl} = y_0\ (2 - K)$
- Near the fixed point write $K^{-1} = 1/2 + t$
- $t\,dt/dl = \pi y_0^2 t$ (1)
- $y_0\ dy_0\ /\mathrm{dl} = 4y_0\ t y_0$ (2)
- 4(1)-π(2), get $2t^2 - \frac{\pi y_0^2}{2} = c$, a constant

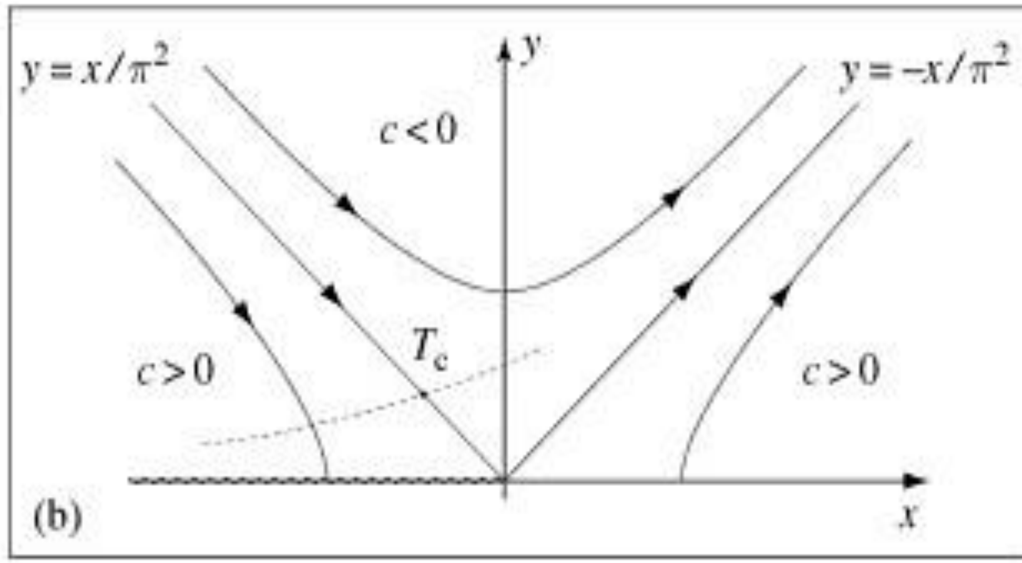
Flow diagram, Different solutions for different c

x=t

$$y^2 = {y_0}^2/4\pi^3$$

$$dy0\ /\mathrm{dl} = 4y0\ t$$

Solid on the left



Fixed point analysis for general systems

- $\frac{dK^{-1}}{d\ln L} = f\left(K^{-1}\right)$. At a fixed point K_c^{-1}, f=0.
- Let $\epsilon = K^{-1} - K_c^{-1}, \epsilon$ is just (T−$T_c)/T_c$, then
- $\frac{d\epsilon}{d\ln L} = f\left(K_c^{-1} + \epsilon\right) = \; g(\epsilon)$, g(0)=0
- Close to the fixed point
- $\frac{dln\,\epsilon}{d\ln L} = y,\; y = g'(0)$. $\epsilon = \epsilon_0 L^y$,
- We also have, for the free energy,
- $F(\epsilon L^y) = L^d F(\epsilon\,)$ for spatial dimension d. This is called the scaling hypothesis.

Magnetic systems

Atoms in magnets are charged particles with angular momenta **S** called spins that, in some units, are interger or half integer.

Their magnetic moments **M** = $g\mu_B$**S** where $g\mu_B$ is some constant.

In a magnetic field **B**, the energy is -**H.M** = -$g\mu_B H$**.S**.

Including heat change, we get the total energy change of a magnetic system is

dE=TdS+other terms-**H.**d**M**

One can define a quantity G=F+BM so that dF'=-SdT+MdH+other terms

Thus $\frac{\partial G}{\partial H} = M$

For processes under a constant magnetic field and at fixed temperature, G is minimized.

Magnetic system,

- For the magnetic system, we have the coupling j=kT/J and H=magnetic field /kT, Similar calculations give, $\epsilon = (\mathrm{T} - T_c)/T_c$
- $G(\epsilon L^y, H\ L^x) = L^d G(\epsilon_, H)$
- This can be written as ($\lambda = L^d, a_\epsilon = \frac{y}{d}, a_H = x/d$)
- $G(\epsilon \lambda^{a_\epsilon}, H\ \lambda^{a_H}) = \lambda G(\epsilon_, H)$
- This is called the static scaling hypothesis.
- From this, we can obtain different the critical exponents and the relationships between them.

- Derivation of critical exponents from the static scaling hypothesis.

Critical exponents

- $G(\epsilon\lambda^{a_\epsilon}, H\ \lambda^{a_H}) = \lambda G(\epsilon, H)$
- Differentiate both sides wrt H
- $\lambda^{a_H}\partial G(\epsilon\lambda^{a_\epsilon}, H\ \lambda^{a_H})/\partial(H\ \lambda^{a_H}) = \lambda\partial G(\epsilon, H)/\partial H$

$$M(\epsilon, 0) \quad \propto (-\epsilon)^{\beta}, \epsilon = (\mathrm{T} - T_c)/T_c$$

- $\lambda^{a_H} M(\epsilon_0 \lambda^{a_\epsilon}, H\, \lambda^{a_H}) = \lambda M(\epsilon_{0,} H)$
- Consider H=0
- $\lambda^{a_H - 1} M(\epsilon \lambda^{a_\epsilon}, 0) = M(\epsilon, 0)$
- Take λ =(-1/ϵ $\}^{1/a_\epsilon}$
- $M(\epsilon, 0) = (-\epsilon)^{(1-a_H)/a_\epsilon} M(-1{,}0) \propto (-\epsilon)^{\beta}$
- $\beta = (1 - a_H)/a_\epsilon$

$$M(0, H) \quad \propto (H)^{1/\delta},$$

- $\lambda^{a_H} M(\epsilon\lambda^{a_\epsilon}, H\,\lambda^{a_H}) = \lambda M(\epsilon, H)$
- Consider ϵ =0, H small
- $\lambda^{a_H - 1} M(0, H\,\lambda^{a_H}) = M(0, H)$
- Take λ =(H}$^{-1/a_H}$
- $M(0, H) = H^{(1-a_H)/a_H} M(0,1) \propto (H)^{\delta}$
- $\delta = a_H/(1 - a_H)$

$$\chi_T(\epsilon, 0) \quad \propto (-\epsilon)^{-\gamma},$$

- $\lambda^{a_H} M(\epsilon\lambda^{a_\epsilon}, H\, \lambda^{a_H}) = \lambda M(\epsilon, H)$
- Magnetic susceptibility at constant T: $\chi_T = \partial M/\partial H$
- $\lambda^{2a_H} \chi_T(\epsilon\lambda^{a_\epsilon}, H\, \lambda^{a_H}) = \lambda \chi_T(\epsilon, H)$
- Take λ =(-1/ϵ $\}^{1/a_\epsilon}$, H=0
- $\chi_T(\epsilon, 0) = (-\epsilon)^{(1-2a_H)/a_\epsilon} \chi_T(-1,0) \propto (-\epsilon)^{-\gamma'}$
- $\gamma' = (2a_H - 1)/a_\epsilon$
- *Recall that* $\delta = a_H/(1 - a_H)$, $\beta = (1 - a_H)/a_\epsilon$
- $\gamma' = \beta(\delta - 1)$

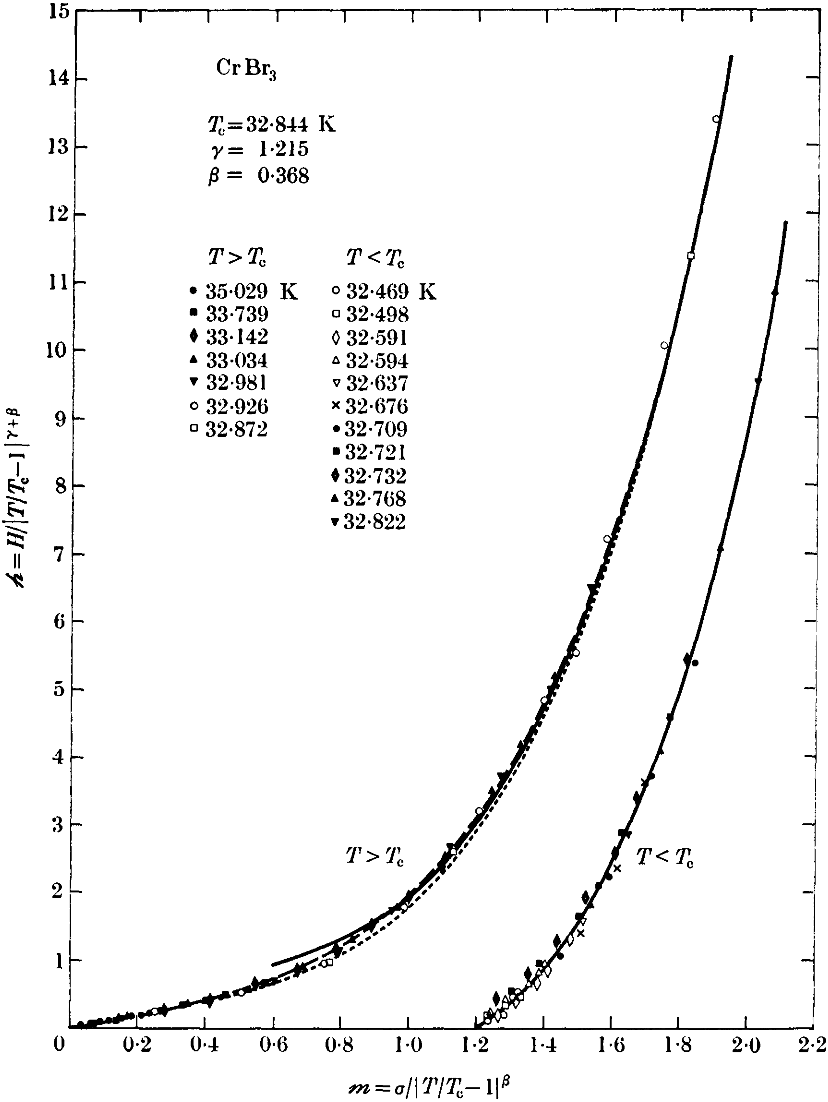


FIG. 11.4. Scaled magnetic field h is plotted against scaled magnetization m for the insulating ferromagnet $CrBr_3$, using data from seven supercritical ($T > T_c$) and from eleven subcritical ($T < T_c$) isotherms. Here $\sigma \equiv M/M_0$. After Ho and Litster (1969).

Note that the determination of the values of two of the exponents is not sufficient to check the validity of the scaling predictions; we need at least three exponents. Of course, if we assume the validity of the scaling hypothesis, (11.30), then determination of two exponents

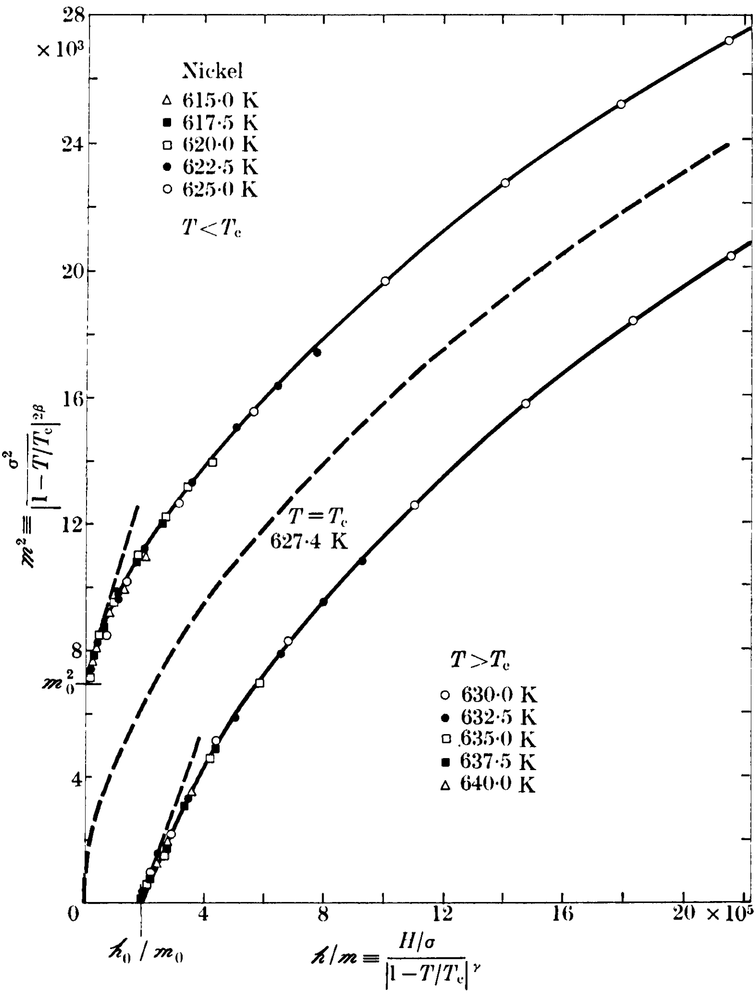


FIG. 11.5. A plot of m^2 against h/m, where m is the scaled magnetization and h is the scaled magnetic field. The data are from measurements on the metallic ferromagnet nickel. Here $\sigma \equiv M/M_0$. After Kouvel and Comly (1968).

suffices to fix the values of all the remaining exponents. For example, the reader can easily verify from Table 11.1 that for $CrBr_3$ the data of (11.64) together with the scaling assumption imply that

$$\left.\begin{aligned} \alpha &\simeq 0{\cdot}05, \\ \Delta &\simeq 1{\cdot}6, \\ \varphi &\simeq 0{\cdot}03, \\ \psi &\simeq 0{\cdot}60. \end{aligned}\right\} \qquad (11.66)$$