2020 · 9 · 30

$\Lambda_n = [-n,n]^2$, ξ_n.

$$\lim_n \phi^{\xi_n}_{p,q,\Lambda_n}[A] = \phi_{p,q}[A]$$

for any event A depending only on finitely many edges.

Prop. $\phi^1_{p,q}$.

1st. A increasing. finitely many edges.

$$\{\phi^1_{p,q,\Lambda_n}[A]\}_n \downarrow.$$

$$P(A) := \lim_n \phi^1_{p,q,\Lambda_n}[A].$$

2nd. A finitely many edges.

$$P(A) := \lim_n \phi^1_{p,q,\Lambda_n}[A].$$

3rd. $P(\cdot)$. can be extended to a proba. measure. $\phi^1_{p,q}$.

Lemma. $q \geq 1$, $p \in [0,1]$. ϕ^0, ϕ^1 translation-inv. ergodic.

Pf: only prove for ϕ^1

$$\phi^1[A] = \lim_n \phi^1_{\Lambda_n}[A]$$

$\Lambda_n = [-n, n]^2$

(1) translation-inv.

suffices to show $\phi^1[A] = \phi^1[\tau_x A]$

for any increasing event A depending on finitely many edges and $|x|=1$.

$$\phi^1[A] = \lim_n \phi^1_{\Lambda_n}[A]$$

$$= \lim_n \phi^1_{\tau_x \Lambda_n}[\tau_x A]$$

$$\Lambda_{n-1} \subset \tau_x \Lambda_n \subset \Lambda_{n+1}.$$

$$\phi^1_{\tau_x \Lambda_n}[\tau_x A] = \phi^1_{\tau_x \Lambda_n}\left[\phi^1_{\tau_x \Lambda_n}[\tau_x A \mid \omega(e)_{e \in \tau_x \Lambda_n \setminus \Lambda_{n-1}}]\right]$$

$$\leq \phi^1_{\Lambda_{n-1}}[\tau_x A]$$

$$\phi'[A] = \lim_n \phi'_{\Lambda_n}[A]$$

$$= \lim_n \phi'_{\tau_x\Lambda_n}[\tau_x A]$$

$$\Lambda_{n-1} \subset \tau_x \Lambda_n \subset \Lambda_{n+1}$$

$$\phi'_{\Lambda_{n+1}}[\tau_x A] \leq \phi'_{\tau_x\Lambda_n}[\tau_x A] \leq \phi'_{\Lambda_{n-1}}[\tau_x A]$$

$n \to \infty$. $\quad \phi'[A] = \phi'[\tau_x A]$.

(2) ergodic. A translation-inv. $\phi'[A] = 0$ or 1.

Claim: B, C depend only on finitely many edges,

$$\lim_{|x|\to\infty} \phi'[B \cap \tau_x C] = \phi'[B]\,\phi'[C].$$

$\forall \varepsilon > 0$. $\exists$ event B depending on finitely many edges

$$\phi'[A \triangle B] \leq \varepsilon.$$

$$\phi'[A] = \phi'[A \cap \tau_x A] = \phi'[B \cap \tau_x B] + O(\varepsilon)$$

$$\underset{|x|\to\infty}{\overset{\text{(claim)}}{=}} \phi'[B]\,\phi'[\tau_x B] + O(\varepsilon)$$

$$= \phi'[A]\,\phi'[\tau_x A] + O(\varepsilon)$$

$$\phi'[A] = \phi'[A]^2 + O(\varepsilon).$$

$\varepsilon \to 0.$ $\phi'[A] = \phi'[A]^2.$ $\phi'[A] = 0$ or 1.

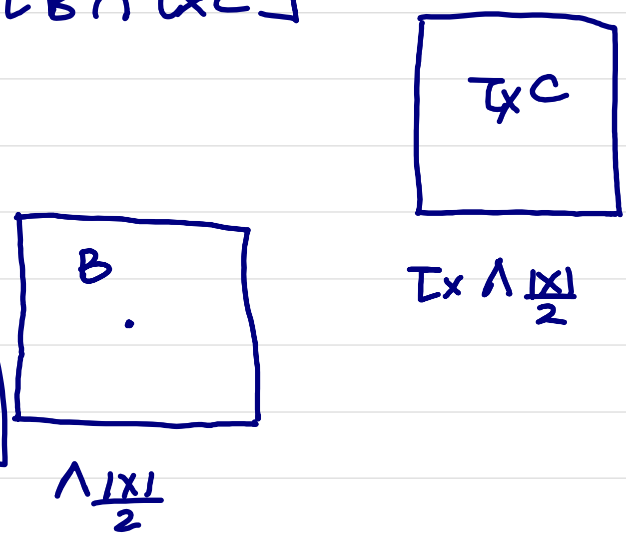
Pf. of Claim: assume B, C increasing. // Λ_N.

$$\phi'[B \cap T_x C] \overset{\text{FKG}}{\geq} \phi'[B]\,\phi'[T_x C]$$

$$= \phi'[B]\,\phi'[C].$$

$$\phi'[B \cap T_x C] = \lim_{n} \phi'_{\Lambda_n}[B \cap T_x C]$$

$n \gg |x| \gg N$



$$\phi'_{\Lambda_n}[B \cap T_x C]$$

$$= \phi'_{\Lambda_n}\left[\phi'_{\Lambda_n}\left[B \cap T_x C \,\middle|\, \begin{matrix} \omega(e) \\ e \in \Lambda_n \setminus \Lambda_{\frac{|x|}{2}} \end{matrix}\right]\right]$$

$$\leq \phi'_{\Lambda_{\frac{|x|}{2}}}[B]\,\phi'_{\Lambda_n}[T_x C]$$

$n \to \infty.$ $|x| \to \infty.$ $\lim_{|x|\to\infty} \phi'[B \cap T_x C] \leq \phi'[B]\,\phi'[C].$

$$\lim_{|x|\to\infty} \phi'[B \cap T_x C] = \phi'[B]\,\phi'[B].$$

Lemma. Fix $p \in [0,1]$, $q \geq 1$.

For $\phi^0_{p,q}$, $\phi^1_{p,q}$. either no ∞-cluster a.s.

or $\exists!$ ∞-cluster a.s.

Pf: ϕ^1. $k \in \{0, 1, 2, \dots, \infty\}$.

$\mathcal{A}_k := \{ \exists\ k\ \infty\text{-clusters} \}$.

translation-inv. ϕ^1 ergodic.

$\phi^1[\mathcal{A}_k] = 0$ or 1.

$\sum_k \phi^1[\mathcal{A}_k] = 1$.

$\exists\, n_0 \in \{0, 1, 2, \dots\}$, $\phi^1[\mathcal{A}_{n_0}] = 1$,

$\phi^1[\mathcal{A}_n] = 0$ for $n \neq n_0$.

$n_0 = 0$, $n_0 = 1$, OK.

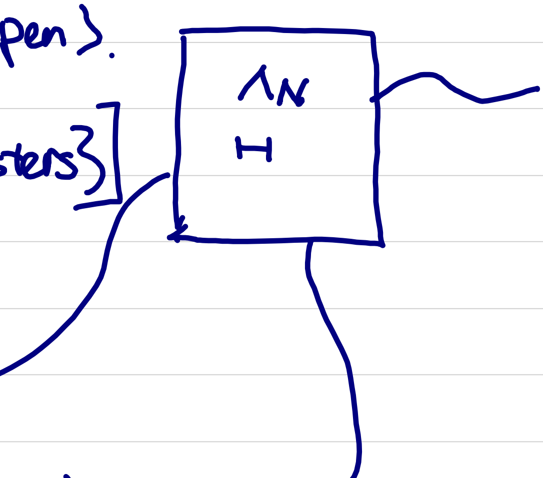
case 1. $2 \leq n_0 < \infty$. case 2. $n_0 = \infty$.

case 1. $2 \le n_0 < \infty$.

$\forall \varepsilon > 0$, $\exists N$ s.t.

$$\phi'[\text{all } n_0\ \infty\text{-clusters intersecting } \Lambda_N] \ge 1-\varepsilon.$$

$E_N = \{\text{all edges in } \Lambda_N \text{ are open}\}$.



$$\phi'[A_1] \ge \phi'[E_N \cap \{\text{all } n_0\ \infty\text{-clusters} \cdots \Lambda_N\}]$$

$$\ge (1-\varepsilon)\cdot\left(\frac{p}{p+(1-p)q}\right)^{\#E(\Lambda_N)} > 0$$

$\phi'[A_1] > 0$. contradiction.

case 2.

replace "independence" by

"finite-energy property".

Thm. Fix $q \geq 1$. $\exists$ $p_c = p_c(q) \in [0,1]$ s.t.

For $p > p_c$, any ∞-volume measure has a unique ∞-cluster a.s.

For $p < p_c$, any ∞-volume measure has no ∞-cluster a.s.

Recall Bernoulli bond perco:

Let $\phi_{p,q}$ be any ∞-volume measure.

$$p_c = \sup\{p : \phi_{p,q}[0 \leftrightarrow \infty] = 0\}.$$

$$p < p_c, \quad \phi_{p,q}[0 \leftrightarrow \infty] = 0$$

$$p > p_c, \quad \phi_{p,q}[0 \leftrightarrow \infty] > 0.$$

Pf of Thm.

$$p_c = \sup\{p : \phi^0_{p,q}[0 \leftrightarrow \infty] = 0\}.$$

$p > p_c$, $\quad \phi_p^0 [0 \leftrightarrow \infty] > 0$.

for any ∞-volume measure ϕ_p,

$$\phi_p [0 \leftrightarrow \infty] \geq \phi_p^0 [0 \leftrightarrow \infty] > 0.$$

$$\phi_p^0 [\exists \infty\text{-cluster}] = 1$$

$$\phi_p [\exists \infty\text{-cluster}] = 1.$$

$p < p_c$ $\quad \phi_p^0 [0 \leftrightarrow \infty] = 0$.

Lemma. $\phi_p^0 = \phi_p^1$ for all but countable many p.

$\tilde{p} \in (p, p_c)$ s.t. $\phi_{\tilde{p}}^0 = \phi_{\tilde{p}}^1$.

for any ∞-volume measure ϕ_p,

$$\phi_p [0 \leftrightarrow \infty] \leq \phi_p^1 [0 \leftrightarrow \infty] \leq \phi_{\tilde{p}}^1 [0 \leftrightarrow \infty]$$

$$= \phi_{\tilde{p}}^0 [0 \leftrightarrow \infty] = 0$$

$$\phi_p [\exists \infty\text{-cluster}] = 0.$$

Lemma. $\phi^0_p = \phi^1_p$ for all but countably many p.

Pf: Claim: $\phi^0[\omega(e)=1] = \phi^1[\omega(e)=1]$ $(\star)$

$(\star)$ implies $\phi^0 = \phi^1$.

suffices to show $\phi^0[A] = \phi^1[A]$

Pf of Claim: for any $\uparrow$ A depending // Λ_N.

consider $\phi^0_{\Lambda_n} \leq_{st} \phi^1_{\Lambda_n}$.

$\exists$ a coupling $\mathbb{P}_n$ (ω_0, ω_1) $\omega_0 \leq \omega_1$.

$\phi^0_{\Lambda_n}$ $\phi^1_{\Lambda_n}$

$$0 \leq \phi^1_{\Lambda_n}[A] - \phi^0_{\Lambda_n}[A]$$

$$= \mathbb{P}_n[\omega_1 \in A] - \mathbb{P}_n[\omega_0 \in A]$$

$$= \mathbb{P}_n[\omega_1 \in A, \omega_0 \notin A]$$

$$\leq \mathbb{P}_n[\exists e \in \Lambda_N, \ \omega_1(e)=1, \ \omega_0(e)=0]$$

$$\leq \sum_{e \in \Lambda_N} \mathbb{P}_n[\omega_1(e)=1, \omega_0(e)=0]$$

$$0 \le \phi^1_{\Lambda_n}[A] - \phi^0_{\Lambda_n}[A]$$

$$\le \sum_{e \in \Lambda_N} \mathbb{P}_n[\omega_1(e)=1,\ \omega_0(e)=0]$$

$$= \sum_{e \in \Lambda_N} \left(\phi^1_{\Lambda_n}[\omega(e)=1] - \phi^0_{\Lambda_n}[\omega(e)=1]\right)$$

$$\xrightarrow[(\star)]{n\to\infty} 0$$

$$\phi^1[A] = \phi^0[A]. \qquad \phi^1 = \phi^0.$$

Pf of Lemma :

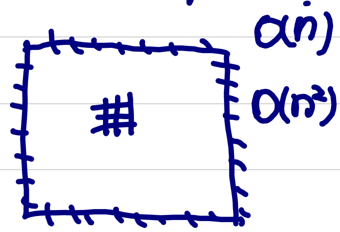
$$\phi^0_p[\omega(e)=1] = \phi^1_p[\omega(e)=1] \qquad (\star)$$

for all but countably many p.

free - energy. $\displaystyle \phi^\xi_{\Lambda_n}[\omega] = \frac{1}{Z^\xi_n} p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}$

$$Z^\xi_n = \sum_\omega p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}$$

$$f(p,q) := \lim_n \frac{1}{\underset{o(n^2)}{\#E(\Lambda_n)}} \log Z^\xi_{p,q,\Lambda_n}.$$



$$Z^{\xi}_{p,q,\Lambda_n} = \sum_{\omega} p^{o(\omega)} (1-p)^{c(\omega)} q^{k(\omega)}$$

$$o(\omega) + c(\omega) = \# E(\Lambda_n)$$

$$= \sum_{\omega} \underbrace{\left(\frac{p}{1-p}\right)}_{} {}^{o(\omega)} q^{k(\omega)} \cdot \underline{(1-p)^{\# E(\Lambda_n)}}$$

define $\pi = \log \frac{p}{1-p}$, $\frac{p}{1-p} = e^{\pi}$.

$$\tilde{f}^{\xi}_n (\pi, q) := \frac{1}{\# E(\Lambda_n)} \log Z^{\xi}_{p,q,\Lambda_n} - \log(1+e^{\pi})$$

$$\downarrow \qquad = \frac{1}{\# E(\Lambda_n)} \log \sum_{\omega} e^{\underline{\pi o(\omega)}} q^{k(\omega)}$$

$$\tilde{f}(\pi, q) = f(p_{\pi}, q) - \log(1+e^{\pi}).$$

$$\partial_{\pi} \tilde{f}^{\xi}_n (\pi, q) = \frac{1}{\# E(\Lambda_n)} \frac{\sum_{\omega} o(\omega) e^{\pi o(\omega)} q^{k(\omega)}}{\sum_{\omega} e^{\pi o(\omega)} q^{k(\omega)}}$$

$$= \frac{1}{\# E(\Lambda_n)} \phi^{\xi}_{p,q,\Lambda_n} [o(\omega)]$$

$$\partial_\pi \tilde{f}_n^{\xi}(\pi, q) = \frac{1}{\# E(\Lambda_n)} \sum_{e \in E(\Lambda_n)} \phi^{\xi}_{p,q,\Lambda_n}[\omega(e)=1]$$

$\partial_\pi \tilde{f}_n^{\xi}(\pi, q)$ increasing in π.

$\tilde{f}_n^{\xi}(\pi, q)$ convex. in π

$\tilde{f}(\pi, q)$ convex in π.

Fact: • (g_n) a sequence of convex functions.

• $g_n \to g$ pointwise.

• g differentiable at x

$\partial^+ g_n(x) \to g'(x)$, $\partial^- g_n(x) \to g'(x)$.

$\tilde{f}(\pi, q)$ convex. differentiable at all but countably many π.

•

Claim: $(\star)$ holds at π where $\tilde{f}(\pi, q)$ is differentiable.

$$\partial_\pi \tilde{f}_n'(\pi, q) = \frac{1}{\# E(\Lambda_n)} \sum_{e \in E(\Lambda_n)} \phi'_{p,q,\Lambda_n}[\omega(e)=1]$$

$$\to \phi'_{p,q}[\omega(e)=1].$$

$$\phi'_{p,q}[\omega(e)=1] = \partial_\pi \tilde{f}(\pi, q).$$

$$\phantom{\phi'_{p,q}[\omega(e)=1] = \partial_\pi} \parallel$$

$$\phi^0_{p,q}[\omega(e)=1] = \partial_\pi \hat{f}(\pi, q).$$