Random hyperbolic and flat surfaces

(Riemann surfaces seen without glasses)

Lecture 8: Method of factorial moments

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Plan for today

We will describe the statistics of very short objects on a random surface of large genus.

Two main results.

- Petri-Mirzakhani: lengths of short closed geodesics on a random hyperbolic surface.
- Masur-Rafi-Randecker: lengths of short saddle connections on a random translation surface.

Common tool: method of factorial moments combined with large genus volume asymptotics.

Poisson distributions and factorial moments

Let N be a random variable with values in $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

We say N is Poisson distributed with mean $\lambda > 0$ if

$$\mathbb{P}(N = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for all } k \in \mathbb{N}_0.$$

Define the *falling factorial of N of order r* by

$$(N)_r = N(N-1)\cdots(N-r+1), \quad r \geq 1.$$

If $\mathbb{E}[(N)_r]$ exists, it is called the rth factorial moment of N.

For Poisson variables these are much simpler than the ordinary moments $\mathbb{E}[N^r]$. We will now compute them explicitly.

Factorial moments of a Poisson variable

Let $N \sim \mathrm{Poisson}(\lambda)$, and fix $r \geq 1$. Then

$$\mathbb{E}[(N)_r] = \sum_{k=0}^{\infty} (k)_r \cdot \mathbb{P}(N = k).$$

For $k < r$ we have $(k)_r = 0$, so

$$\mathbb{E}[(N)_r] = \sum_{k=r}^{\infty} \frac{k!}{(k-r)!} \cdot \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=r}^{\infty} \frac{\lambda^k}{(k-r)!}.$$

Let $m = k - r$. Then

$$\mathbb{E}[(N)_r] = e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+r}}{m!} = e^{-\lambda} \lambda^r \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = e^{-\lambda} \lambda^r e^{\lambda} = \lambda^r.$$

So for all $r \geq 1$,

$$\mathbb{E}[(N)_r] = \lambda^r.$$

Method of factorial moments: one random variable

The method of factorial moments says the inverse is also true.

Theorem (Method of factorial moments, one variable)

Let N_g be random variables on probability spaces $(\Omega_g, \mathbb{P}_g)$ and let $\lambda > 0$. Assume that for every $r \in \mathbb{N}$,

$$\lim_{g \to \infty} \mathbb{E}_g[(N_g)_r] = \lambda^r.$$

Then N_g converges in distribution to a Poisson random variable of mean λ.

So to prove a Poisson limit, it is enough to compute asymptotics of

$$\mathbb{E}_g[N_g(N_g - 1) \cdots (N_g - r + 1)]$$

for all r.

A simple model: sum of independent trials

Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with

$$\mathbb{P}(X_i = 1) = p, \qquad \mathbb{P}(X_i = 0) = 1 - p.$$

Set

$$N = X_1 + \cdots + X_n,$$

so N is the number of successes in n independent trials, that is

$$N \sim \mathrm{Bin}(n, p).$$

We want to understand the factorial moments

$$(N)_r = N(N-1)\cdots(N-r+1),$$

and see why they are much easier to compute than the usual moments $\mathbb{E}[N^r]$.

Combinatorial meaning of $(N)_r$

Fix $r \geq 1$. Expand

$$(N)_r = N(N-1)\cdots(N-r+1)$$

in terms of the X_i. Each X_i is either 0 or 1, hence

$$(N)_r = \sum_{\substack{(i_1,\ldots,i_r) \\ \text{distinct}}} X_{i_1} X_{i_2} \cdots X_{i_r}.$$

because each ordered r–tuple $(i_1, \ldots, i_r)$ of distinct indices contributes 1 if all $X_{i_j} = 1$, and 0 otherwise.

That is, $(N)_r$ is the number of ordered r–tuples of distinct successes among the n trials.

Computing the factorial moments

Take expectations and use linearity:

$$\mathbb{E}[(N)_r] = \mathbb{E}\left[\sum_{\substack{(i_1,\ldots,i_r)\\ \text{distinct}}} X_{i_1}\cdots X_{i_r}\right] = \sum_{\substack{(i_1,\ldots,i_r)\\ \text{distinct}}} \mathbb{E}[X_{i_1}\cdots X_{i_r}].$$

By independence,

$$\mathbb{E}[X_{i_1}\cdots X_{i_r}] = \mathbb{P}(X_{i_1} = \cdots = X_{i_r} = 1) = p^r.$$

The number of ordered r–tuples of distinct indices in $\{1,\ldots,n\}$ is

$$n(n-1)\cdots(n-r+1).$$

Hence

$$\mathbb{E}[(N)_r] = n(n-1)\cdots(n-r+1)\,p^r.$$

From binomial to Poisson via factorial moments

Let $N_n \sim \mathrm{Bin}(n, p_n)$, and suppose

$$n\,p_n \longrightarrow \lambda > 0 \quad \text{as } n \to \infty.$$

For fixed $r \geq 1$,

$$\begin{aligned}\mathbb{E}[(N_n)_r] &= n(n-1)\cdots(n-r+1)\,p_n^r. \\ &= \left(\frac{n(n-1)\cdots(n-r+1)}{n^r}\right)(np_n)^r.\end{aligned}$$

As $n \to \infty$, $np_n \longrightarrow \lambda$ and

$$\frac{n(n-1)\cdots(n-r+1)}{n^r} \longrightarrow 1, \qquad \Longrightarrow \qquad \mathbb{E}[(N_n)_r] \longrightarrow \lambda^r.$$

By the method of factorial moments, N_n converges in distribution to a Poisson random variable with mean λ.

Joint factorial moments

There is also a joint version of this.

Let $(N_{1,g}, \ldots, N_{k,g})$ be a vector of random variables. For a multi index $(r_1, \ldots, r_k)$ set

$$(N_{1,g})_{r_1} \cdots (N_{k,g})_{r_k}.$$

Theorem (Method of moments, joint version)

Let $(N_{1,g}, \ldots, N_{k,g})$ be random vectors taking values in $\mathbb{N}_0^k$. Assume there exist $\lambda_1, \ldots, \lambda_k > 0$ such that for all $r_1, \ldots, r_k \in \mathbb{N}$,

$$\lim_{g\to\infty} \mathbb{E}_g\left[(N_{1,g})_{r_1} \cdots (N_{k,g})_{r_k}\right] = \lambda_1^{r_1} \cdots \lambda_k^{r_k}.$$

Then $(N_{1,g}, \ldots, N_{k,g})$ converges in distribution to a vector of independent Poisson variables with means $(\lambda_1, \ldots, \lambda_k)$.

Random hyperbolic surfaces and Weil-Petersson volumes

For $g \geq 0$ and $n \geq 0$, let $\mathcal{M}_{g,n}(L)$ be the moduli space of hyperbolic surfaces of genus g with n geodesic boundary components of lengths $L = (L_1, \ldots, L_n)$.

The Weil–Petersson volume of $\mathcal{M}_{g,n}(L)$ is denoted by

$$V_{g,n}(L) = \mathrm{Vol}_{WP}(\mathcal{M}_{g,n}(L)).$$

Mirzakhani showed that $V_{g,n}(L)$ is a polynomial in $L_1^2, \ldots, L_n^2$ with coefficients given by intersection numbers of Chern classes of tautological line bundles.

These polynomials satisfy recursion relations that allow one to compute all volumes in principle and to study their asymptotics as g tends to infinity.

Key Volume Estimates

Theorem (Mirzakhani)

For any fixed $n \geq 0$:

$$\frac{V_{g-1,n+2}}{V_{g,n}} = 1 + \frac{3-2n}{\pi^2} \cdot \frac{1}{g} + O\Big(\frac{1}{g^2}\Big).$$

Theorem (Mirzakhani)

Let $g, n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}_+$ be small compared to g. Then

$$\frac{V_{g,n}(2x_1, \ldots, 2x_n)}{V_{g,n}} = \prod_{i=1}^{n} \frac{\sinh(x_i)}{x_i} \cdot \left(1 + O\Big(\frac{1}{g}\Big)\right).$$

Counting short closed geodesics

Fix $0 \leq a < b$. For a hyperbolic surface $X \in \mathcal{M}_g$ let

$N_{g,[a,b]}(X) = \big|\{\gamma \text{ primitive closed geodesic on } X \mid \ell_X(\gamma) \in [a,b]\}\big|.$

So $N_{g,[a,b]}$ is a random variable on $(\mathcal{M}_g, \mathbb{P}_g)$, where

$$\mathbb{P}_g(A) = \frac{1}{V_g} \int_A d\,\mathrm{Vol}_{WP}\,.$$

We are interested in the joint behavior of

$$N_{g,[a_1,b_1]}, \ldots, N_{g,[a_k,b_k]}$$

for disjoint intervals $[a_i, b_i]$.

The Petri-Mirzakhani theorem

Define
$$\lambda[a,b] = \int_a^b \frac{e^t + e^{-t} - 2}{2t}\,dt.$$

Theorem (Petri and Mirzakhani)

Let $[a_1,b_1],\ldots,[a_k,b_k] \subset \mathbb{R}_+$ be disjoint intervals. Then, as $g \to \infty$, the random vector
$$\left(N_{g,[a_1,b_1]},\ldots,N_{g,[a_k,b_k]}\right)$$
converges in distribution to a vector of independent Poisson random variables with means $\lambda[a_i,b_i]$.

So the bottom part of the length spectrum of a random hyperbolic surface looks like a Poisson point process on $(0,\infty)$ with intensity
$$\frac{e^t + e^{-t} - 2}{2t}\,dt.$$

Strategy of Petri–Mirzakhani

Fix an interval $[a, b] \subset (0, \infty)$. We split moduli space into two parts:

$$\mathcal{M}_g = \mathcal{M}_g^{\mathrm{gen}} \sqcup \mathcal{M}_g^{\mathrm{exc}}.$$

On the **generic part** $\mathcal{M}_g^{\mathrm{gen}}$ every geodesic of length in $[a, b]$ is:

- simple,
- disjoint from any other such geodesic, and
- nonseparating.

On the **exceptional part** $\mathcal{M}_g^{\mathrm{exc}}$ at least one of these fails (intersections, separating curves, too many short curves, etc.).

They show:

$$\mathbb{E}_g[(N_{g,[a,b]})_r] = \mathbb{E}_g\left[(N_{g,[a,b]})_r \mathbf{1}_{\mathcal{M}_g^{\mathrm{gen}}}\right] + o(1)$$

and compute the main term using Mirzakhani's volume formulas.

First factorial moment on the generic set

Start with $r = 1$ on the generic set. By Mirzakhani's integration formula, the contribution of nonseparating curves to the expectation can be written as

$$\mathbb{E}_g\left[N_{g,[a,b]}\mathbf{1}_{\mathcal{M}_g^{\mathrm{gen}}}\right] = \frac{1}{V_{g,0}}\int_a^b \Phi_g(t)\,dt + O\Big(\frac{1}{g}\Big),$$

where

$$\Phi_g(t) = \frac{t}{2}\,V_{g-1,2}(t,t).$$

We now use the two key volume estimates:

$$\frac{V_{g-1,2}}{V_{g,0}} = 1+O\Big(\frac{1}{g}\Big), \qquad \frac{V_{g-1,2}(t,t)}{V_{g-1,2}} = \Big(\frac{\sinh(t/2)}{t/2}\Big)^2\Big(1+O\Big(\frac{1}{g}\Big)\Big),$$

where in the second line we applied

$$\frac{V_{g-1,2}(2x_1,2x_2)}{V_{g-1,2}} = \frac{\sinh(x_1)}{x_1}\frac{\sinh(x_2)}{x_2}\Big(1+O\Big(\frac{1}{g}\Big)\Big)$$

with $x_1 = x_2 = t/2$.

Computing the intensity for $r = 1$

We have

$$\frac{\Phi_g(t)}{V_{g,0}} = \frac{t}{2} \cdot \frac{V_{g-1,2}(t,t)}{V_{g-1,2}} \cdot \frac{V_{g-1,2}}{V_{g,0}} = \frac{t}{2} \cdot \Big(\frac{\sinh(t/2)}{t/2}\Big)^2 \Big(1 + O\Big(\frac{1}{g}\Big)\Big).$$

A short computation:

$$\begin{aligned}\frac{t}{2}\Big(\frac{\sinh(t/2)}{t/2}\Big)^2 &= \frac{t}{2} \cdot \frac{4\sinh^2(t/2)}{t^2} = \frac{2}{t}\,\sinh^2(t/2)\\ &= \frac{2}{t}\,\frac{e^t + e^{-t} - 2}{4} = \frac{e^t + e^{-t} - 2}{2t}.\end{aligned}$$

So

$$\frac{\Phi_g(t)}{V_{g,0}} = \frac{e^t + e^{-t} - 2}{2t}\Big(1 + O\Big(\frac{1}{g}\Big)\Big),$$

and hence

$$\mathbb{E}_g\big[N_{g,[a,b]}\mathbf{1}_{\mathcal{M}_g^{\mathrm{gen}}}\big] = \int_a^b \frac{e^t + e^{-t} - 2}{2t}\,dt + O\Big(\frac{1}{g}\Big) = \lambda[a,b] + O\Big(\frac{1}{g}\Big).$$

Higher factorial moments on the generic set

For $r \geq 1$, on the generic set $\mathcal{M}_g^{\mathrm{gen}}$ an ordered r–tuple counted by

$$(N_{g,[a,b]})_r = N(N-1)\dots(N-r+1)$$

is exactly an ordered r–tuple of disjoint, simple, nonseparating primitive curves, each of length in $[a,b]$.

Mirzakhani's integration formula then gives

$$\mathbb{E}_g\left[(N_{g,[a,b]})_r \mathbf{1}_{\mathcal{M}_g^{\mathrm{gen}}}\right] = \frac{1}{V_{g,0}} \int_{[a,b]^r} \Phi_g^{(r)}(t_1,\dots,t_r)\, dt_1 \cdots dt_r + O\Big(\frac{1}{g}\Big),$$

where, for nonseparating configurations,

$$\Phi_g^{(r)}(t_1,\dots,t_r) = \frac{t_1 \cdots t_r}{2^r}\, V_{g-r,2r}(t_1,\dots,t_r).$$

Higher factorial moments, continued

Applying the multi–boundary volume estimate with $x_i = t_i/2$,

$$\frac{V_{g-r,2r}(t_1,\ldots,t_r)}{V_{g-r,2r}} = \prod_{i=1}^{r}\Big(\frac{\sinh(t_i/2)}{t_i/2}\Big)^2\Big(1+O\Big(\frac{1}{g}\Big)\Big),$$

and

$$\frac{V_{g-r,2r}}{V_{g,0}} = 1+O\Big(\frac{1}{g}\Big),$$

we obtain

$$\frac{\Phi_g^{(r)}(t_1,\ldots,t_r)}{V_{g,0}} = \prod_{i=1}^{r}\frac{t_i}{2}\Big(\frac{\sinh(t_i/2)}{t_i/2}\Big)^2\Big(1+O\Big(\frac{1}{g}\Big)\Big).$$

Each factor equals $\frac{e^{t_i}+e^{-t_i}-2}{2t_i}$, as in the $r=1$ computation.

Factorial moments and the Poisson limit

Integrating over $[a,b]^r$ we get

$$\begin{aligned}\mathbb{E}_g\left[(N_{g,[a,b]})_r \mathbf{1}_{\mathcal{M}_g^{\mathrm{gen}}}\right] &= \int_{[a,b]^r} \prod_{i=1}^{r} \frac{e^{t_i}+e^{-t_i}-2}{2t_i}\, dt_1 \cdots dt_r + O\Big(\frac{1}{g}\Big) \\ &= \Big(\int_a^b \frac{e^{t}+e^{-t}-2}{2t}\, dt\Big)^r + O\Big(\frac{1}{g}\Big) \\ &= \lambda[a,b]^r + O\Big(\frac{1}{g}\Big).\end{aligned}$$

On the exceptional set $\mathcal{M}_g^{\mathrm{exc}}$ the Weil–Petersson volume is $o(V_{g,0})$ and the contribution to the factorial moments is $o(1)$. Hence

$$\lim_{g\to\infty} \mathbb{E}_g[(N_{g,[a,b]})_r] = \lambda[a,b]^r \quad \text{for all } r \geq 1.$$

The same argument with several disjoint intervals gives a vector of independent Poisson variables.

An application: the systole

Let $\mathrm{sys}(X)$ be the length of the shortest closed geodesic on X.

Then

$$\mathbb{P}_g\big(\mathrm{sys}(X) \leq \epsilon\big) = \mathbb{P}_g\big(N_{g,[0,\epsilon]} \geq 1\big) = 1 - \mathbb{P}_g\big(N_{g,[0,\epsilon]} = 0\big).$$

By Petri and Mirzakhani's theorem,

$$\lim_{g\to\infty} \mathbb{P}_g\big(\mathrm{sys}(X) \leq \epsilon\big) = 1 - e^{-\lambda[0,\epsilon]} \sim \frac{\epsilon^2}{2} \quad \text{as } \epsilon \to 0.$$

In particular, the distribution of the systole converges and the expected systole tends to a positive constant.

Random translation surfaces in the principal stratum

Let H_g be the principal stratum of area one translation surfaces of genus g, that is

$$H_g = H_g(1, \ldots, 1),$$

where there are $2g-2$ simple zeros.

We endow H_g with the Masur–Veech measure μ_{MV}. Its total volume

$$\mathrm{Vol}(H_g) = \mu_{MV}(H_g)$$

is finite. We consider the probability measure

$$\mathbb{P}_g(A) = \frac{1}{\mathrm{Vol}(H_g)} \int_A d\mu_{MV}.$$

A *random translation surface* of genus g is an element $(X, \omega) \in H_g$ chosen according to $\mathbb{P}_g$.

Counting short saddle connections

Let $\ell_\omega(\alpha)$ denote the flat length of a saddle connection α on (X, ω). As mentioned in the last class, different features of the surface appear at different scales.

It turns out the first saddle connections appear at the scale $1/g$. Hence, we define

$$N_{g,[a,b]}(X, \omega) = \left|\left\{\text{saddle connections } \alpha \,\middle|\, \ell_\omega(\alpha) \in [a/g, b/g]\right\}\right|.$$

Again, for disjoint intervals $[a_i, b_i] \subset \mathbb{R}_+$ we consider the vector

$$N_{g,[a_1,b_1]}, \ldots, N_{g,[a_k,b_k]}.$$

The analogue of the Petri-Mirzakhani theorem is: these counts converge to independent Poisson variables, but now with a different intensity function.

The Masur-Rafi-Randecker theorem

Theorem (Masur, Rafi and Randecker)

Let $[a_1, b_1], \ldots, [a_k, b_k] \subset \mathbb{R}_+$ *be disjoint intervals. Then, as* $g \to \infty$*, the random vector*

$$\left(N_{g,[a_1,b_1]}, \ldots, N_{g,[a_k,b_k]}\right) : H_g \to \mathbb{N}_0^k$$

converges in distribution to a vector of independent Poisson variables with means

$$\lambda[a_i, b_i] = 8\pi(b_i^2 - a_i^2), \quad i = 1, \ldots, k.$$

In particular, the expected number of saddle connections of length in $[a/g, b/g]$ *on a random surface in* H_g *tends to* $8\pi(b^2 - a^2)$*.*

Consequences for the shortest saddle connection

Let $\ell_{\min}(X, \omega)$ be the length of the shortest saddle connection on (X, ω).

Then

$$\mathbb{P}_g\big(\ell_{\min}(X, \omega) < \epsilon/g\big) = 1 - \mathbb{P}_g\big(N_{g,[0,\epsilon]} = 0\big).$$

By the theorem,

$$\lim_{g\to\infty} \mathbb{P}_g\big(\ell_{\min} < \epsilon/g\big) = 1 - e^{-\lambda[0,\epsilon]} = 1 - e^{-8\pi\epsilon^2} \sim 8\pi\epsilon^2 \quad \text{as } \epsilon \to 0.$$

So the typical shortest saddle connection has length on the order of $1/g$ and its distribution has an explicit limit law.

Generic and non generic subsets of H_g

Fix $0 < a_1 < b_1 \leq \cdots \leq a_k < b_k$ and let $B = \max_i b_i$.

Non generic set $H_{NG} \subset H_g = \mathcal{H}_g(1, \ldots, 1)$:

- surfaces with a loop (saddle connection from a zero to itself) of length at most $\frac{B}{g}$, or
- surfaces with two short saddle connections of length at most $\frac{B}{g}$ sharing a zero (including short homologous saddle connections).

Define the generic set by

$$H_{Gen} = H_g \smallsetminus H_{NG}.$$

Using Siegel–Veech constants one shows

$$\frac{\mathrm{Vol}(H_{NG})}{\mathrm{Vol}(H_g)} = O\Big(\frac{1}{g}\Big), \qquad \frac{\mathrm{Vol}(H_{Gen})}{\mathrm{Vol}(H_g)} \longrightarrow 1.$$

So we can first do all factorial moment calculations on H_{Gen}.

The combinatorial cover $\widehat{H}$

Fix integers $r_1, \ldots, r_k \geq 1$ and put

$$K = r_1 + \cdots + r_k.$$

For each $(X, \omega) \in H_{Gen}$, consider ordered lists

$$\Gamma_i = (\alpha_{i,1}, \ldots, \alpha_{i,r_i})$$

of disjoint *oriented* saddle connections with lengths in the interval

$$\left[\frac{a_i}{g}, \frac{b_i}{g}\right].$$

Let $\widehat{H}$ be the set of tuples

$$\widehat{H} = \{(X, \omega, \Gamma) \mid (X, \omega) \in H_{Gen},\ \Gamma = (\Gamma_1, \ldots, \Gamma_k)\}.$$

Simultaneous collapsing

On H_{Gen} we can *simultaneously collapse* all these saddle connections to obtain:

$$\Phi : \widehat{H} \to M_{r_1,\dots,r_k} \times H' \times \prod_{i=1}^{k} A^{r_i}_{a_i,b_i},$$

where:

- $H' = H_g(2,\dots,2,1,\dots,1)$ has K zeros of order 2 and the rest of order 1,
- $M_{r_1,\dots,r_k}$ encodes the combinatorics of which zeros are paired,
- $A_{a_i,b_i} \subset \mathbb{R}^2$ is the annulus with inner radius a_i/g and outer radius b_i/g.

The map Φ is:

- locally measure preserving in period coordinates, and
- 3^K–to–1 (each double zero can be opened in 3 different ways).

Local Jacobian and volume of $\widehat{H}$

The area of an annulus A_{a_i,b_i} is

$$\mathrm{Area}(A_{a_i,b_i}) = \pi\left(\frac{b_i^2}{g^2} - \frac{a_i^2}{g^2}\right) = \frac{\pi}{g^2}\,(b_i^2 - a_i^2).$$

One obtains the volume estimate (for large g):

$$\mathrm{Vol}(\widehat{H}) = 3^K\ \mathrm{Vol}(H'_{Gen})\left(1+K\cdot O(1/g)\right)\cdot\prod_{i=1}^{k}\left[\frac{\pi(b_i^2 - a_i^2)}{g^2}\right]^{r_i}\cdot|M_{r_1,\dots,r_k}|.$$

Here $|M_{r_1,\dots,r_k}|$ is the number of ways to choose and label the endpoints of the K saddle connections:

$$|M_{r_1,\dots,r_k}| = \frac{(2g-2)!}{(2g-2-2K)!}.$$

Ratio of volumes: how many double zeros?

We now compare the volume of the auxiliary stratum H' to that of the principal stratum:

$$H' = H_g(2,\ldots,2,1,\ldots,1), \qquad H_g = H_g(1,\ldots,1).$$

By the large genus asymptotics of Masur–Veech volumes (Aggarwal, Chen–Möller–Sauvaget–Zagier and others), we have

$$\frac{\mathrm{Vol}(H')}{\mathrm{Vol}(H_g)} = \left(\frac{4}{3}\right)^K \cdot \left(1 + O\left(\frac{1}{g}\right)\right),$$

and the same holds with H_{Gen} and H'_{Gen} in place of H_g and H':

$$\frac{\mathrm{Vol}(H'_{Gen})}{\mathrm{Vol}(H_{Gen})} = \left(\frac{4}{3}\right)^K \cdot \left(1 + O\left(\frac{1}{g}\right)\right).$$

Intuitively: each time we turn two simple zeros into one double zero, the volume grows by a factor asymptotic to $4/3$.

Factorial moments on the generic set

Recall

$$Y_{g,r_1,\ldots,r_k} = (N_{g,[a_1,b_1]})_{r_1} \cdots (N_{g,[a_k,b_k]})_{r_k}$$

counts ordered lists of saddle connections in the given windows.

On H_{Gen}, each choice of K oriented saddle connections gives rise to 2^K different choices of orientations for Γ, hence

$$\mathbb{E}_{Gen}\big(Y_{g,r_1,\ldots,r_k}\big) = \frac{1}{\mathrm{Vol}(H_{Gen})} \cdot \frac{1}{2^K} \int_{\widehat{H}} 1 \, d\mu.$$

Using the volume of $\widehat{H}$ we just computed, we get

$$\mathbb{E}_{Gen}\big(Y_{g,r_1,\ldots,r_k}\big) = \frac{\mathrm{Vol}(H'_{Gen})}{\mathrm{Vol}(H_{Gen})} \cdot \frac{3^K}{2^K} \cdot \frac{(2g-2)!}{(2g-2-2K)!} \cdot \prod_{i=1}^{k} \left[\frac{\pi(b_i^2 - a_i^2)}{g^2}\right]^{r_i} \cdot \Big(1 + K \cdot O(1/g)\Big).$$

Extracting the constant $8\pi(b_i^2 - a_i^2)$

As $g \to \infty$, we have

$$\frac{\mathrm{Vol}(H'_{Gen})}{\mathrm{Vol}(H_{Gen})} \to \left(\frac{4}{3}\right)^K .$$

Next, the combinatorial factor satisfies

$$\frac{(2g-2)!}{(2g-2-2K)!} \sim (2g)^{2K}.$$

Putting everything together:

$$\lim_{g\to\infty} \mathbb{E}_{Gen}\left(Y_{g,r_1,\dots,r_k}\right) = \left(\frac{4}{3}\right)^K \cdot \frac{3^K}{2^K} \cdot (2g)^{2K} \cdot \prod_{i=1}^{k} \left(\frac{\pi(b_i^2 - a_i^2)}{g^2}\right)^{r_i}$$

$$= \prod_{i=1}^{k} \left(8\pi(b_i^2 - a_i^2)\right)^{r_i}.$$

Thus, for each i,

$$\lambda[a_i, b_i] = 8\pi(b_i^2 - a_i^2)$$

is the limiting factorial moment parameter.

From factorial moments to the length spectrum

For disjoint intervals $[a_i, b_i]$, the mixed factorial moments factorize:

$$\lim_{g\to\infty} \mathbb{E}_{Gen}\left[(N_{g,[a_1,b_1]})_{r_1} \cdots (N_{g,[a_k,b_k]})_{r_k}\right] = \prod_{i=1}^{k} \left(8\pi(b_i^2 - a_i^2)\right)^{r_i}.$$

The method of factorial moments then implies:

- For each fixed interval $[a_i, b_i]$, $N_{g,[a_i,b_i]}$ converges in law to Poisson with parameter $\lambda[a_i, b_i] = 8\pi(b_i^2 - a_i^2)$.
- For disjoint intervals, the limiting Poisson variables are independent.

Finally, one checks that replacing H_{Gen} by all of H_g does not change the limit, since $\mathrm{Vol}(H_{NG})/\,\mathrm{Vol}(H_g) \to 0$.

Similarities between hyperbolic vs flat

Both use the method of factorial moments and Poisson approximation.

Both rely on precise large genus asymptotics of volumes of moduli spaces.

In both cases the dominant contribution comes from very simple configurations of disjoint short objects.

The method should work whenever these conditions are present. For example:

- Are there similar Poisson laws for quadratic differentials, in particular for the principal stratum $\mathcal{Q}_g(1, 1, ..., 1)$?
- Can one describe the joint limit of short closed geodesics in $\mathcal{H}(1, 1, ..., 1)$?

Thank you