Conformal Invariance in 2D Lattice Models

Part 2: Random Cluster Model

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Part 1: Bernoulli Percolation
Part 2: Random Cluster Model
Part 3: Ising Model

Bernoulli percolation vs. FK-percolation

Bernoulli percolation

Independent percolation

- FKG inequality
- Phase transition
- Critical value : $p_c = p_{sd}$
- Subcritical : exp. decay
- Continuity of PT

FK percolation

dependent percolation

- True for $q \geq 1$
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- True for $1 \leq q \leq 4$.
- False for $q > 4$

FK-percolation—definition

Fortuin and Kasteleyn

FK-percolation : also called random-cluster model. It is a generalization of Bernoulli percolation where there is dependence between edges.

- $G = (V, E)$ is a finite graph
- configuration $\omega \in \{0, 1\}^E$, $o(\omega)$, $c(\omega)$, $k(\omega)$
- edge-parameter $p \in [0, 1]$, cluster-parameter $q > 0$

FK-percolation on G is the probability measure defined by

$$\phi_{p,q,G}[\omega] \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}.$$

FK-percolation—boundary conditions

Fix a partition ξ of ∂G, and identify the vertices in ∂G that belong to the same component of ξ. FK-percolation on G with parameters (p, q) and boundary conditions ξ is the probability measure :

$$\phi^{\xi}_{p,q,G}[\omega] \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}.$$

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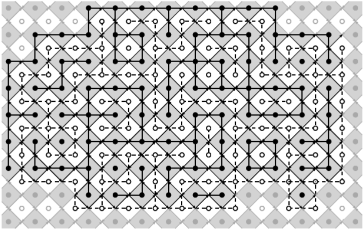
- wired-b.c. : $\phi^{1}_{p,q,G}$
- free-b.c. : $\phi^{0}_{p,q,G}$
- Dobrushin-b.c.
- induced by a config. outside G



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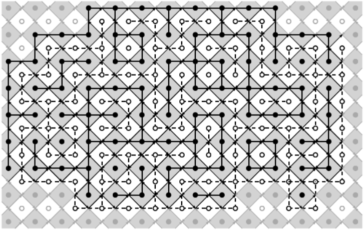

Domain Markov Property

Suppose $G' \subset G$, for any $\psi \in \{0,1\}^{E(G)\setminus E(G')}$,

$$\phi^{\xi}_{p,q,G}[X \,|\, \omega_e = \psi_e, \forall e \in E(G) \setminus E(G')] = \phi^{\psi^{\xi}}_{p,q,G'}[X].$$

Theorem (FKG Inequality)

Fix $p \in [0,1]$, $q \geq 1$, a finite graph G and some boundary conditions ξ. For any two increasing events A and B, we have

$$\phi^{\xi}_{p,q,G}[A \cap B] \geq \phi^{\xi}_{p,q,G}[A]\phi^{\xi}_{p,q,G}[B].$$

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- Given two proba. measures μ_1, μ_2, we write $\mu_1 \leq_{st} \mu_2$, if $\mu_1[A] \leq \mu_2[A]$ for all increasing event A.
- A proba. measure μ strictly positive if $\mu[\omega] > 0$ for all ω.

Theorem (Holley inequality)

Let μ_1, μ_2 be strictly positive probability measures on the finite state space such that

$$\mu_2[\omega^e]\mu_1[\eta_e] \geq \mu_2[\omega_e]\mu_1[\eta^e], \quad \forall e \in E, \forall \eta \leq \omega.$$

Then $\mu_1 \leq_{st} \mu_2$.

FKG Inequality : consequences

Corollary (Monotonicity)

Fix $p \le p'$ and $q \ge 1$, a finite graph G and some b.c. ξ.
We have $\phi^{\xi}_{p,q,G} \le_{st} \phi^{\xi}_{p',q,G}$.

Corollary (Comparison between boundary conditions)

Fix $p \in [0,1]$ and $q \ge 1$, a finite graph G. For any b.c. $\xi \le \psi$,
we have $\phi^{\xi}_{p,q,G} \le_{st} \phi^{\psi}_{p,q,G}$.
In particular, for any b.c. ξ, we have $\phi^{0}_{p,q,G} \le_{st} \phi^{\xi}_{p,q,G} \le_{st} \phi^{1}_{p,q,G}$.

Corollary (Finite-energy property)

Fix $p \in [0,1]$ and $q \ge 1$, a finite graph G, and some b.c. ξ, we have

$$\frac{p}{p+(1-p)q} \le \phi^{\xi}_{p,q,G}[\omega(f)=1 \,|\, \omega(e)=\psi(e)\ \forall e \in E(G)\setminus\{f\}] \le p.$$

Bernoulli percolation vs. FK-percolation

Bernoulli percolation	FK percolation
Independent percolation	dependent percolation
FKG inequality	True for $q \geq 1$ ✓
Phase transition	True for $q \geq 1$
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Infinite volume measure

Let ξ_n be a sequence of b.c. The sequence $\phi^{\xi_n}_{p,q,\Lambda_n}$ is said to converge to the infinite-volume measure $\phi_{p,q}$ if

$$\lim_n \phi^{\xi_n}_{p,q,\Lambda_n}[A] = \phi_{p,q}[A],$$

for any event A depending only on the status of finitely many edges.

Proposition

Fix $p \in [0,1]$ and $q \geq 1$. There exist two (possibly equal) infinite-volume random-cluster measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ such that for any event A depending on a finite number of edges,

$$\lim_{n\to\infty} \phi^1_{p,q,\Lambda_n}[A] = \phi^1_{p,q}[A], \quad \lim_{n\to\infty} \phi^0_{p,q,\Lambda_n}[A] = \phi^0_{p,q}[A].$$

Ergodicity

Lemma

Fix $q \geq 1$. *The infinite-volume measures* $\phi^0_{p,q}$ *and* $\phi^1_{p,q}$ *are translation invariant and are ergodic.*

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Lemma

Fix $q \geq 1$. *For* $\phi^0_{p,q}$ *or* $\phi^1_{p,q}$, *either there is no infinite cluster almost surely, or there exists a unique infinite cluster almost surely.*

Ergodicity

Lemma

Fix $q \geq 1$. The infinite-volume measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are translation invariant and are ergodic.

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Fix $q \geq 1$. For $\phi^0_{p,q}$ or $\phi^1_{p,q}$, either there is no infinite cluster almost surely, or there exists a unique infinite cluster almost surely.

The measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are extremal :

$$\phi^0_{p,q} \leq_{st} \phi_{p,q} \leq_{st} \phi^1_{p,q}.$$

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$$\phi^0_{p,q} \leq_{st} \phi_{p,q} \leq_{st} \phi^1_{p,q}.$$

Question

Do we have $\phi^0_{p,q} = \phi^1_{p,q}$?

Bernoulli percolation vs. FK-percolation

Bernoulli percolation Independent percolation	FK percolation dependent percolation
FKG inequality	True for $q \geq 1$ ✓
Phase transition	$q \geq 1$: ∞-volume measure ✓ $q \geq 1$: Phase transition
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Phase transition

Theorem

Fix $q \geq 1$. There exists a critical point $p_c = p_c(q) \in [0, 1]$ such that

- *For $p > p_c$, any infinite-volume measure has an infinite cluster almost surely.*
- *For $p < p_c$, any infinite-volume measure has no infinite cluster almost surely.*

Lemma

Fix $q \geq 1$. we have $\phi^0_{p,q} = \phi^1_{p,q}$ for all but countably many values of p.

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Critical Value : self-dual point

Theorem

Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \geq 1$. The critical value p_c is given by

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Proposition

The dual configuration of the random-cluster model on G with parameters (p, q) and b.c. ξ is the random-cluster model with parameters (p^*, q) on G^* with b.c. ξ^* where $p^* = p^*(p, q)$ satisfying

$$\frac{pp^*}{(1-p)(1-p^*)} = q.$$

Critical Value

Lemma

Fix $q \geq 1$, we have

$$\phi^0_{p_{sd}(q),q}[0 \leftrightarrow \infty] = 0.$$

Theorem

Consider the random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \geq 1$.

- *If $p < p_c$, then there exists $c = c(p) > 0$ such that for every $n \geq 1$, $\phi^1_{p,q,\Lambda_n}[0 \longleftrightarrow \partial\Lambda_n] \leq e^{-cn}$.*
- *If $p > p_c$, then there exists $C > 0$ such that $\phi^1_{p,q}[0 \longleftrightarrow \infty] \geq C(p - p_c)$.*

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• Phase transition	• $q \geq 1$: ∞-volume measure ✓ • $q \geq 1$: Phase transition ✓
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Continuity of the phase transition

Theorem

- *Fix* $1 \le q \le 4$, *we have*

$$\phi^1_{p_c,q}[0 \longleftrightarrow \infty] = 0.$$

- *Fix* $q > 4$, *we have*

$$\phi^1_{p_c,q}[0 \longleftrightarrow \infty] > 0, \quad \phi^0_{p_c,q}[0 \longleftrightarrow \infty] = 0$$

Consequence

- When $1 \le q \le 4$, we have $\phi^1 = \phi^0$, and continuous PT.
- When $q > 4$, we have $\phi^1_{p_c,q} \neq \phi^0_{p_c,q}$, and discontinuous PT for $\phi^1_{p_c,q}$.

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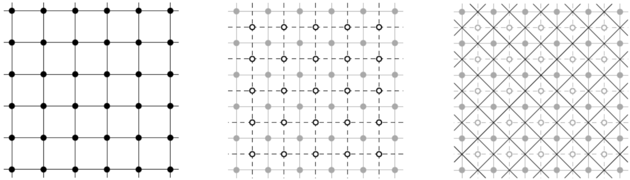
FK-Ising model

FK-Ising model : random-cluster model with $q = 2$.



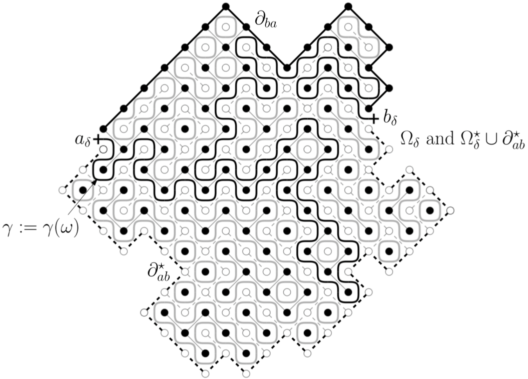
(a) The square lattice. (b) The dual lattice. (c) The medial lattice.

$\mathbb{L} = (\mathbb{Z}^2, E(\mathbb{Z}^2))$: the square lattice ; $\mathbb{L}^*$: the dual lattice
$\mathbb{L}^\diamond$: the medial lattice.

- vertices : the centers of edges of $\mathbb{L}$.
- edges : connecting nearest neighbors.

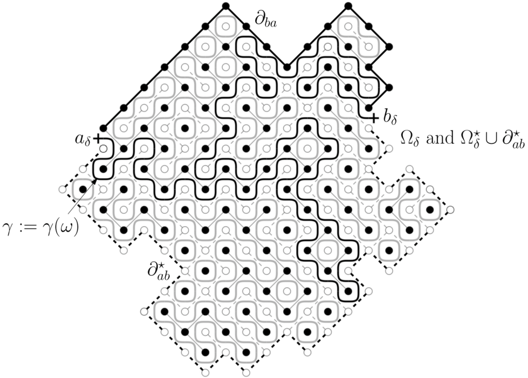
$\mathbb{L}_\delta = \sqrt{2}\delta\mathbb{L}$, $\mathbb{L}_\delta^*$, $\mathbb{L}_\delta^\diamond$. The mesh-size of $\mathbb{L}_\delta^\diamond$ is δ.

Dobrushin domain



For a simply connected domain Ω, we set $\Omega_\delta = \Omega \cap \mathbb{L}_\delta$.
For a Dobrushin domain $(\Omega; a, b)$, let $(\Omega_\delta^\diamond; a_\delta, b_\delta)$ be an approximation.
Dobrushin b.c. : edges of (ba) are open (edges of (ba) are wired), edges of $(a^* b^*)$ are dual-open (edges of (ab) are free).

Loop representation



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Draw self-avoiding loops on $\Omega^\diamond$ as follows : a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open edges through this vertex.

FK fermionic observable

Definition

The edge FK fermionic observable is defined on edges of $\Omega_\delta^\diamond$ by

$$F_{(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)}(e) = \mathbb{E}_{(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)}\left[1_{\{e\in\gamma\}} \exp\left(\frac{i}{2} W_\gamma(e, b_\delta)\right)\right],$$

where $W_\gamma(e, b_\delta)$ denotes the winding between the center of e and $b_\delta^\diamond$. The vertex FK fermionic observable is defined on vertices of $\Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond$ by

$$F_{(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)}(v) = \frac{1}{2}\sum_{e\sim v} F_{(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)}(e),$$

where the sum is over the four medial edges having v as an endpoint.

Conformal invariance

Theorem

Fix a Dobrushin domain $(\Omega; a, b)$. Consider the critical FK-Ising model. Let F_δ be the vertex fermionic observable in $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$. Then, we have

$$\frac{1}{\sqrt{2\delta}} F_\delta \to \sqrt{\phi'}, \quad \text{as } \delta \to 0, \quad \text{locally uniformly,}$$

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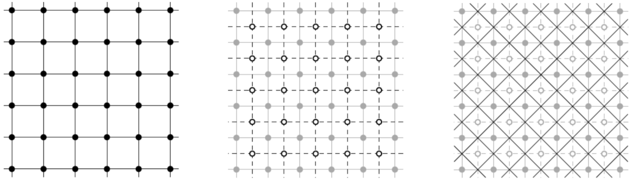
Discrete complex analysis



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Discrete complex analysis

For $x \in \mathbb{L}_\delta$ and $h : \mathbb{L}_\delta \to \mathbb{C}$, define

$$\Delta_\delta h(x) = \frac{1}{4} \sum_{y:\ y \sim x} (h(y) - h(x)).$$

- $h : \Omega_\delta \to \mathbb{C}$ is preharmonic if $\Delta_\delta h(x) = 0, \forall x \in \Omega_\delta$.
- $h : \Omega_\delta \to \mathbb{C}$ is pre-superharmonic if $\Delta_\delta h(x) \leq 0, \forall x \in \Omega_\delta$.
- $h : \Omega_\delta \to \mathbb{C}$ is pre-subharmonic if $\Delta_\delta h(x) \geq 0, \forall x \in \Omega_\delta$.

The classical relation between preharmonic function and SRW :
Let (X_n) be a SRW on $\mathbb{L}_\delta$ killed at the first time it exits Ω_δ, then h is preharmonic if and only if $(h(X_n))$ is a martingale.

Convergence of the discrete Dirichlet problem solution

Theorem

- *let* $(\Omega; a, b)$ *be a Dobrushin domain,*
- *let* f *be a bounded continuous function on* $\partial\Omega \setminus \{a, b\}$*,*
- *let* h *be the unique harmonic function on* Ω*, continuous on* $\overline{\Omega} \setminus \{a, b\}$*, satisfying* $h = f$ *on* $\partial\Omega \setminus \{a, b\}$*.*
- *let* $(\Omega_\delta; a_\delta, b_\delta)$ *be a sequence of discrete Dobrushin domains converging to* $(\Omega; a, b)$ *in the Carathéodory sense.*
- *let* $f_\delta : \partial\Omega_\delta \to \mathbb{C}$ *be a sequence of uniformly bounded functions converging to* f *uniformly away from* a *and* b*.*
- *let* h_δ *be the unique preharmonic function on* Ω_δ *such that* $h_\delta = f_\delta$ *on* $\partial\Omega_\delta$*.*

Then $h_\delta \to h$ *locally uniformly as* $\delta \to 0$*.*

Preholomorphic function

For a function $f : \mathbb{L}_\delta \to \mathbb{C}$, and $x \in \mathbb{L}_\delta^*$, define

$$\bar{\partial}_\delta f(x) = \frac{1}{2}(f(E) - f(W)) + \frac{i}{2}(f(N) - f(S)),$$

where N, E, S, W are the four vertices of $\mathbb{L}_\delta$ adjacent to x indexed in the obvious way.
A function $f : \Omega_\delta \to \mathbb{C}$ is preholomorphic if $\bar{\partial}_\delta f(x) = 0$ for all $x \in \Omega_\delta^*$.
The equation $\bar{\partial}_\delta f(x) = 0$ is called the Cauchy-Riemann equation at x.

(1) Sums of preholomorphic functions are preholomorphic.

(2) Discrete contour integrals vanish in simply connected domain.

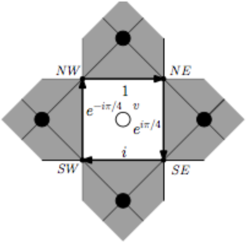
(3) The primitive in simply connected domain is well-defined.

(4) If a family (f_δ) of preholomorphic functions on Ω_δ converges locally uniformly to f on Ω, then f is holomorphic.

Attention : the product of two preholomorphics is not preholomorphic.

Spin-holomorphic

For $e \in E(\mathbb{L}^\diamond)$, we give an orientation : counterclockwise around white faces.
For $e \in E(\mathbb{L}^\diamond)$, we associate a direction $\ell(e)$: as in the figure. In other words, $\ell(e)$ has the same direction as $\sqrt{\bar{e}}$.



A function f is s-holomorphic if for any edge e of $\Omega_\delta^\diamond$, we have

$$P_{\ell(e)}[f(x)] = P_{\ell(e)}[f(y)],$$

where x, y are the endpoints of e and P_ℓ is the orthogonal projection on the direction ℓ.

Proposition

Any s-holomorphic function $f : \Omega_\delta^\diamond \to \mathbb{C}$ is preholomorphic on $\Omega_\delta^\diamond$.

Discrete analog of $\frac{1}{2}\Im \int f^2$

Theorem

Let Ω be a simply connected domain. Suppose $f : \Omega_\delta^\diamond \to \mathbb{C}$ is an s-holomorphic function and $b_0 \in \Omega_\delta$. Then, there exists a unique function $H : \Omega_\delta \cup \Omega_\delta^* \to \mathbb{C}$ such that

$$H(b_0) = 1, \quad \textit{and} \quad H(b) - H(w) = \delta |P_{\ell(e)}[f(x)]|^2 (= \delta |P_{\ell(e)}[f(y)]|^2),$$

for every edge $e = (x, y)$ on $\Omega_\delta^\diamond$ bordered by a black face $b \in \Omega_\delta$ and a white face $w \in \Omega_\delta^*$.

For two neighboring sites $b_1, b_2 \in \Omega_\delta$, with v being the medial vertex at the center of (b_1, b_2),

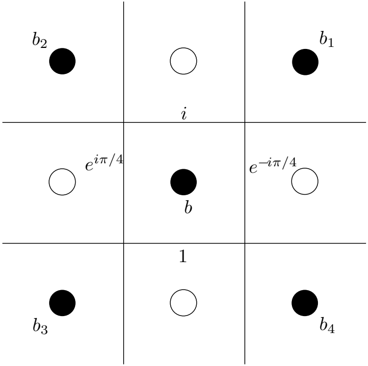
$$H(b_1) - H(b_2) = \frac{1}{2} \Im (f(v)^2 (b_1 - b_2)).$$

The same relation also holds for vertices of Ω_δ^*.

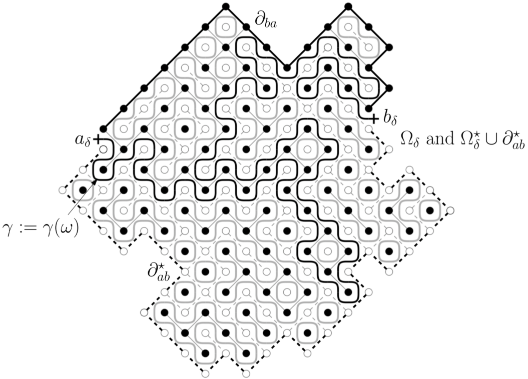
Discrete analog of $\frac{1}{2}\Im \int f^2$

Proposition

Denote by $H^\bullet$ the restriction of H to Ω_δ (black faces) and by H° the restriction of H to Ω_δ^* (white faces). If f is s-holomorphic, then $H^\bullet$ is subharmonic and H° is superharmonic.

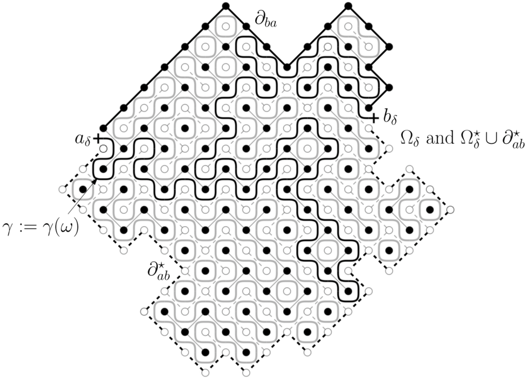


Dobrushin domain



For a simply connected domain Ω, we set $\Omega_\delta = \Omega \cap \mathbb{L}_\delta$.
For a Dobrushin domain $(\Omega; a, b)$, let $(\Omega_\delta^\diamond; a_\delta, b_\delta)$ be an approximation.
Dobrushin b.c. : edges of (ba) are open (edges of (ba) are wired), edges of $(a^* b^*)$ are dual-open (edges of (ab) are free).

Loop representation



Fix a Dobrushin domain $(\Omega; a, b)$ with Dobrushin b.c.
Draw self-avoiding loops on $\Omega^\diamond$ as follows : a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open edges through this vertex.

FK fermionic observable

Definition

The edge FK fermionic observable is defined on edges of $\Omega_\delta^\diamond$ by

$$F_{(\Omega_\delta^\diamond;a_\delta^\diamond,b_\delta^\diamond)}(e)=\mathbb{E}_{(\Omega_\delta^\diamond;a_\delta^\diamond,b_\delta^\diamond)}\left[1_{\{e\in\gamma\}}\exp\left(\frac{i}{2}W_\gamma(e,b_\delta)\right)\right],$$

where $W_\gamma(e,b_\delta)$ denotes the winding between the center of e and $b_\delta^\diamond$.
The vertex FK fermionic observable is defined on vertices of $\Omega_\delta^\diamond\setminus\partial\Omega_\delta^\diamond$ by

$$F_{(\Omega_\delta^\diamond;a_\delta^\diamond,b_\delta^\diamond)}(v)=\frac{1}{2}\sum_{e\sim v}F_{(\Omega_\delta^\diamond;a_\delta^\diamond,b_\delta^\diamond)}(e),$$

where the sum is over the four medial edges having v as an endpoint.

Conformal invariance

Theorem

Fix a Dobrushin domain $(\Omega; a, b)$. Consider the critical FK-Ising model. Let F_δ be the vertex fermionic observable in $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$. Then, we have

$$\frac{1}{\sqrt{2\delta}} F_\delta \to \sqrt{\phi'}, \quad \text{as } \delta \to 0, \quad \text{locally uniformly},$$

where ϕ is any conformal map from Ω on to the strip $\mathbb{R} \times (0, 1)$ sending a to $-\infty$ and b to $+\infty$.

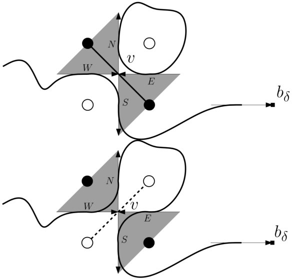
The observable is s-holomorphic

Lemma

Consider a medial vertex $v \in \Omega_\delta^\diamond \setminus \partial\Omega_\delta^\diamond$. We have

$$F_\delta(N) - F_\delta(S) = i\,(F_\delta(E) - F_\delta(W))\,,$$

where N, E, S, W are the four adjacent edges indexed in clockwise order.



Lemma

The vertex fermionic observable F_δ is s-holomorphic.

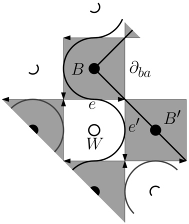
Let A be the black face bordering a_δ. Define $H_\delta : \Omega_\delta \cup \Omega_\delta^* \to \mathbb{R}$ such that

$$H(A) = 1, \quad \text{and} \quad H_\delta(B) - H_\delta(W) = |P_{\ell(e)}[F_\delta(x)]|^2 = |P_{\ell(e)}[F_\delta(y)]|^2,$$

for the medial edge $e = (x, y)$ bordered by a black face $B \in \Omega_\delta$ and a white face $W \in \Omega_\delta^*$.

Lemma

- *The subharmonic function $H_\delta^\bullet$ is equal to 1 on (ba), and it converges to 0 on (ab) uniformly away from a and b.*
- *The superharmonic function H_δ° is equal to 0 on $(a^* b^*)$, and it converges to 1 on $(b^* a^*)$ uniformly away from a and b.*



Proposition

The sequence $(H_\delta)_{\delta>0}$ converges to $\Im\phi$ locally uniformly.

Theorem

Fix a Dobrushin domain $(\Omega; a, b)$*. Consider the critical FK-Ising model. Let* F_δ *be the vertex fermionic observable in* $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$*. Then, we have*

$$\frac{1}{\sqrt{2\delta}} F_\delta \to \sqrt{\phi'}, \quad \text{as } \delta \to 0, \quad \text{locally uniformly},$$

where ϕ *is any conformal map from* Ω *on to the strip* $\mathbb{R} \times (0, 1)$ *sending a to* $-\infty$ *and b to* $+\infty$*.*

Corollary

The exploration path in FK-Ising with Dobrushin boundary conditions converges to $\mathrm{SLE}_{16/3}$. (Lecture on Oct. 30th)