4-dimensional topology

Topics:

- classical invariants of 4-mfds
- Freedman's work
- Constructing 4-manifolds (Kirby calculus)
- Seiberg-witten invariants and applications
  - Donaldson's diagonalizability theorem
  - Furuta's theorem
  - Thom conjecture, Milnor conjecture
- Khovanove homology and exotic $\mathbb{R}^4$s

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- Further topics (e.g. Gabai's light bulb theorem)

Time: Wednesday / Friday 9:50 - 11:25 AM

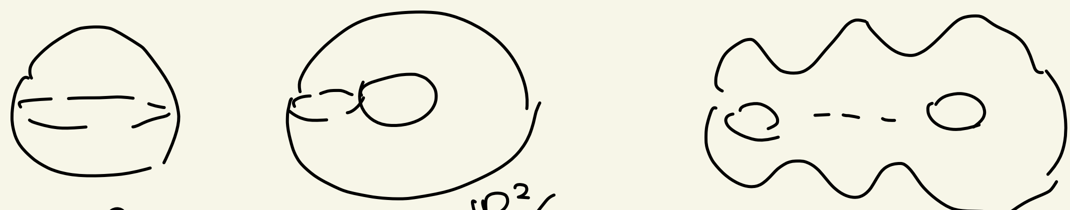
Office Hours: TBD

Lecture 1: Why is dimension 4 special?

Q: Given n, classify connected, closed n-dim mfd up to homeomorphism / diffeomorphism. ($\cong_{top}$ / $\cong_{diff}$)

n=1 $M = S^1$

n=2 $M \begin{cases} \text{orientable, genus } g \geq 0 \\ \text{nonorientable} \quad g \geq 1 \end{cases}$



S^2 $\quad T^2 = \mathbb{R}^2/\mathbb{Z}^2$ $\quad \Sigma_g = \mathbb{H}^2/\Gamma$

n=3 Thurston's geometrization conjecture

$M \xrightarrow{\text{cut along } S^2, T^2, \text{ Klein bottle}} M_1, \cdots, M_i$

Each M_i admits one of the eight geometries

I.e. $M_i = \tilde{M}_i/\Gamma$, $\tilde{M}_i$ a simply-connected homogeneous space ($\tilde{M}_i = G/K$) and $\Gamma \subset G$.

$n \geq 4$ No such decomposition into geometric pieces is known.

Actually, a complete classification is impossible

Theorem (Markov) Given any finite presentation of a group $G = \langle g_1, \dots, g_m \mid r_1, \dots, r_m \rangle$, one can construct a n-dim ($n \geq 4$) mfd M with $\pi_1(M) = G$.

Theorem (Adyan-Rubin) $\nexists$ an algorithm that tells whether a given presentation yields the trivial group.

Proof of Markov's theorem: Take

$M_0 = (S^1 \times S^{n-1}) \# \cdots \# (S^1 \times S^{n-1}) \quad \pi_1(M_0) = \langle g_1, \dots g_m \mid \emptyset \rangle$

Each $r_i \subset \pi_1(M_0)$ is represented by an embeded loop $\gamma_i \subset M_0$.

Surgery along γ_i: $(M_0 - \nu(\gamma_i)) \underset{S^1 \times S^{n-2}}{\cup} (D^2 \times S^{n-2})$

M = surgery on M_0 along $\gamma_1, \dots, \gamma_n$.

$n \geq 4 \Rightarrow \pi_1(S^1 \times S^{n-2}) = \pi_1(D^2 \times D^{n-2}) = 1$

Van-Kampen $\Rightarrow \pi_1(M) = {\pi_1(M_0)}/{N(r_1, \dots, r_n)} = G$. $\square$

Q: Given $n \geq 4$, classify n-dim mfd M with $\pi_1(M) = 1$.

Exotic phenomina: smooth category $\neq$ topological cat.

C = smooth / topological / Piecewise linear category

Let M_0, M_1 be oriented C-manifold. A cobordism W from M_0 to M_1 is a $(n+1)$-dim C-manifold s.t. $\partial W = M_0 \sqcup \overline{M_1}$. We say W is h-cobordism if $M_0 \hookrightarrow W$ $M_1 \hookrightarrow M$ are homotopy equivalences.

Theorem (Smale, Kirby-Siebenman) Let W be an h-cobordism (in C) from M_0 to M_1. Suppose $\pi_1(W) = 1$, $\dim(M_0) \geq 5$. Then $W \cong_C M_0 \times [0,1]$.

proof: handle decomposition, Morse theory

will discuss later.

There is a generalization when $\pi_1 \neq 1$. "S-cobordism" theorem (Barry Mazur, John Stallings, Dennis Barden)

Key Lemma (Whitney) Let F_0, F_1 be connected, orientable manifolds smoothly embeded in M. Suppose

I. $\dim F_0 + \dim F_1 = \dim M \geq 5$

II. $\pi_1(M) = \pi_1(M \setminus F_0) = \pi_1(M \setminus F_1) = 1$

Then we can isotope F_0, F_1 s.t. they intersect transversely at $|F_0 \cdot F_1|$ points.

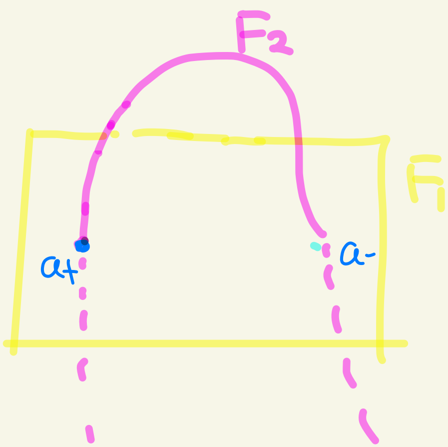
$(F_0 \cdot F_1 := (P.D.[F_0] \cup P.D.[F_1])[M])$

proof: Isotope F_0, F_1 s.t. they intersect transversely.

Suppose we have a positive/negative intersection point a_+/a_- in $F_0 \cap F_1$.

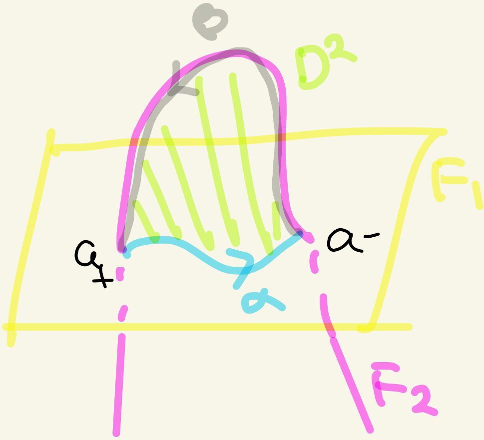
Draw simple arcs $F_1 \supset \alpha : a_+ \rightsquigarrow a_-$

$F_2 \supset \beta : a_- \rightsquigarrow a_+$



$\alpha \circ \beta$: loop in M_1. By I+II $\pi_1(M_1 - F_1 - F_2) = 0$ so $\alpha \circ \beta$ bounds a D^2 whose interior doesn't intersect F_1, F_2.

$\dim M \geq 5 > 2+2 \Rightarrow$ we can isotope D^2 s.t. D^2 is embeded



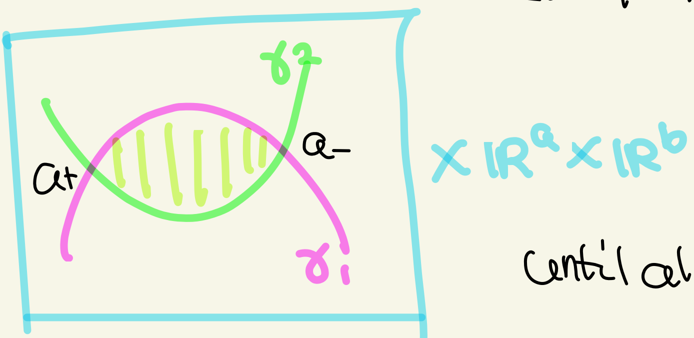
In a regular neighborhood of D_2, we can identify

$M \rightsquigarrow \mathbb{R}^2 \times \mathbb{R}^a \times \mathbb{R}^b$

$F_1 \rightsquigarrow \gamma_1 \times \mathbb{R}^a \times \{0\}$

$F_2 \rightsquigarrow \gamma_2 \times \{0\} \times \mathbb{R}^b$

Isotope F_1 to $\gamma_1' \times \mathbb{R}^a \times \{0\}$.



This cancels a_+, a_-.

Keep doing this until all intersection points are of the same sign. $\square$.

Generalized Poincaré conjecture:

Let M be a n-dim topological manifold homotopy equivalent to S^n. Then $M \simeq_{top} S^n$.

$n \geq 5$ Smale, Stallings, Zeeman, Newman

$n=4$ Freedman

$n=3$ Perelman

Corollary of h-cobordism theorem:

Let M be a smooth n-dim mfd homotopy equiv to S^n. Then $M \simeq_{top} S^n$.

Proof: Remove two small balls $W = M \setminus (\mathring{D}_1^n \cup \mathring{D}_2^n)$

Then W is an h-cobordism from S^{n-1} to S^{n-1}

By h-cobordism theorem $\exists$ diffeomorphism $f: W \to S^{n-1} \times I$

Define a homeomorphism

$$\tilde{f}: M = D_1^n \cup W \cup D_2^n \longrightarrow S^n = D_1^n \times (S^{n-1} \times I) \times D_2^n$$

$$x \in W \longmapsto f(x)$$

$$r \cdot x \in D_1^n \longmapsto r \cdot f(x) \quad r \in [0,1],\ x \in \partial D_1^n \subset W$$

$$r \cdot x \in D_2^n \longmapsto r \cdot f(x) \quad r \in [0,1],\ x \in \partial D_2^n \subset W \quad \square$$

Note: generalized Poincaré conjecture is not true in smooth category for a generalized n.

$\Theta_n^n :=$ {smooth mfds homotopy equiv to S^n}

$= \{$smooth structures on $S^n\}$ smooth h-cobordism

Theorem (Kervarie-Milnor) $n \geq 5$. Θ_n^n can be expressed in terms of $\pi_{n+m}(S^m)$ $m \gg 0$.

Now we know S^{2k+1} has unique smooth str.

$\Longleftrightarrow 2k+1 = 1, 3, 5, 61 \leftarrow$ (Wang-Xu 2016)

In dimension 4, no such connection is known.

Conjecture (SPC4): S^4 has a unique smooth str.

There are possible counter examples. We don't have suitable invariants. (Recent years: Khovanov homology?)

Theorem (Freedman) In topological category, h-cobordism holds in dimension 4.

$\rightsquigarrow$ a classification of simply connected topological 4-mfds.

In smooth category, things are very mysterious.

Theorem (Wall) Let M_0, M_1 be simply-connected smooth 4-mfd. Suppose $M_0 \simeq_{\text{homotopy}} M_1$. Then

(1) M_0 is smoothly h-cobordant to M_1

(2) For $m \gg 0$ $M_0 \#^m (S^2 \times S^2) \simeq_{\text{diff}} M_1 \#^m (S^2 \times S^2)$

Theorem (Donaldson) $\exists$ Smooth 4-mfd M s.t.

$M \simeq_{\text{top}} \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ $\quad$ $M \not\simeq_{\text{diff}} \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$.

Cor: In smooth category, the h-cobordism fails in dimension 4.

Smooth invariants:

Donaldson's polynomial invariants $\{D_k : H^2(X;\mathbb{R}) \to \mathbb{R}\}_{k \geq 0}$

Seiberg-Witten invariants $SW : H^2(X;\mathbb{Z}) \to \mathbb{Z}$

Exotic phenomina in dimension 4 is special:

- $n \neq 4$, $\mathbb{R}^n$ admits a unique smooth structure (Stallings)
- $\mathbb{R}^4$ has uncountable many smooth structure (Donaldson, Gompf, Taubes)
- $n \neq 4$, n-dim mfd M admits only finitely many smooth structures. (Kirby-Siebenman)
- $n = 4$, many known examples admits infinitely many smooth structures.
  (all known examples of irreducible 4-mfd with $b_2^+(X) > 1$.)