Random hyperbolic and flat surfaces
(Riemann surfaces seen without glasses)

Lecture 3. Count of flat closed geodesics and of saddle connections. Siegel–Veech formula

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Teichmüller Theorem. Teichmüller geodesic flow

Flat metric associated to meromorphic quadratic differential

Construction 1. In a simply-connected coordinate chart $\mathcal{U}$, on a Riemann surface, in which a meromorphic quadratic differential $q(w) = \phi(w) \cdot (dw)^2$ does not have zeroes and poles, it can be represented as a square of a non-vanishing holomorphic 1-form: $q(w) = (\pm\omega(w))^2 = (\pm\sqrt{\phi(w)}dw)^2$. The form ω is defined up to a sign.

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The form ω is defined up to a sign.
We can choose a local coordinate z in $\mathcal{U}$ such that in this coordinate $\omega = dz$.
The coordinate $z = x + iy$ is defined up to an additive constant. It is called the *flat coordinate* associated to q. It defines a flat metric coming from the standard Euclidean plane endowed with coordinates x, y and *horizontal* ($y = const$) and *vertical* ($x = const$) foliations, which are orthogonal in our flat metric.
Neither the flat metric nor the foliations depend on a choice of sign of $\omega = dz$.

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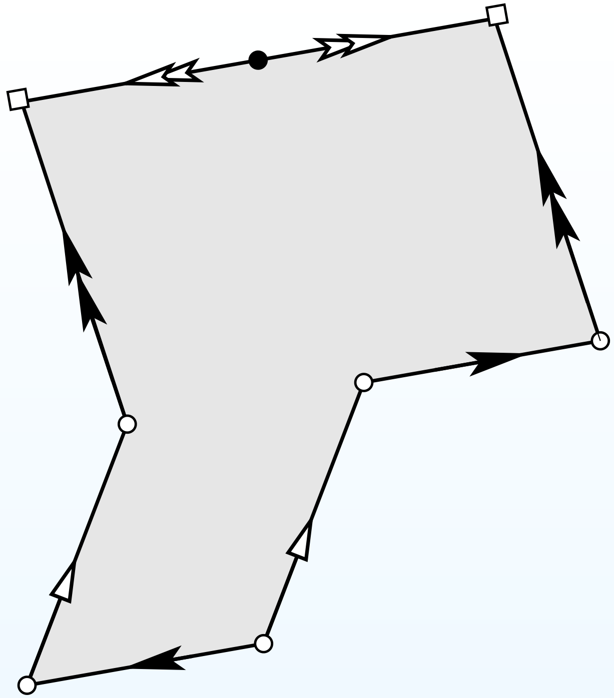
Construction 2. Let $w = u + iv$. Define a volume element in $\mathcal{U}$ as

$$-\frac{1}{2i}|\phi(w)|\, dw \wedge d\bar{w} = |\phi(w)|\, du \wedge dv$$

and a length element as $\sqrt{|\phi(w)|}\,|dw| = \sqrt{|\phi(w)|}\sqrt{du^2 + dv^2}$.

Exercise. Verify that the two constructions of the flat metric are equivalent and that the resulting metric does not depend on coordinates.

Half-translation surfaces

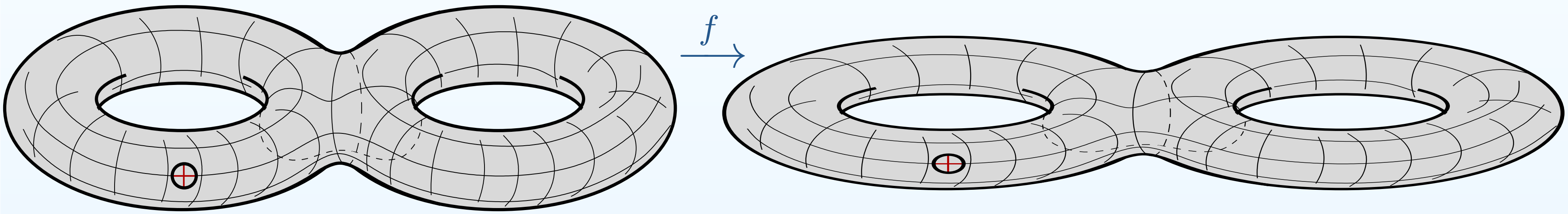


As in the case of Abelian differentials one can unwrap the resulting flat surface to a polygon. This time the sides are identified not only by parallel translations, but also by central symmetries. We still have distinguished vertical and horizontal directions. The difference with Abelian differentials is that now the holonomy group of the metric is $\mathbb{Z}/2\mathbb{Z}$: a parallel transport along a smooth loop can bring a tangent vector $\vec{v}$ back to itself or to $-\vec{v}$.

We can let the meromorphic quadratic differential have simple poles. They correspond to cone angles π of the metric. When the poles are at most simple, the area of the surface is still finite.

Coefficient of quasiconformality

Let X_1 and X_2 be Riemann surfaces of genus g. When complex structures are different there are no conformal maps from X_1 to X_2. A smooth map $f : X_1 \to X_2$ sends an infinitesimal circle at $x \in X_1$ to an infinitesimal ellipse at $f(x)$.



Coefficient of quasiconformality of f at $x \in X_1$ is the ratio $K_x(f) = \frac{a}{b}$ of demi-axis of this ellipse. *Coefficient of quasiconformality* of f is

$$K(f) = \sup_{x \in X_1} K_x(f).$$

Though X_1 is a compact Riemann surface we use $\sup$ and not $\max$ since the map f is allowed to have several isolated points where f is degenerate or even not smooth, and where $K_x(f)$ is thus not defined.

Teichmüller Theorem

Theorem. *For any pair X_1, X_2 of Riemann surfaces of genus $g \geq 1$ there exist an extremal map $f_{ext} : X_1 \to X_2$ which minimizes the coefficient of quasiconformality $K(f)$. For this extremal map f_{ext} the coefficient of quasiconformality is constant everywhere on X_1 outside of a finite collection of points, where f_{ext} degenerates, and where $K_x(f_{ext})$ is not defined.*

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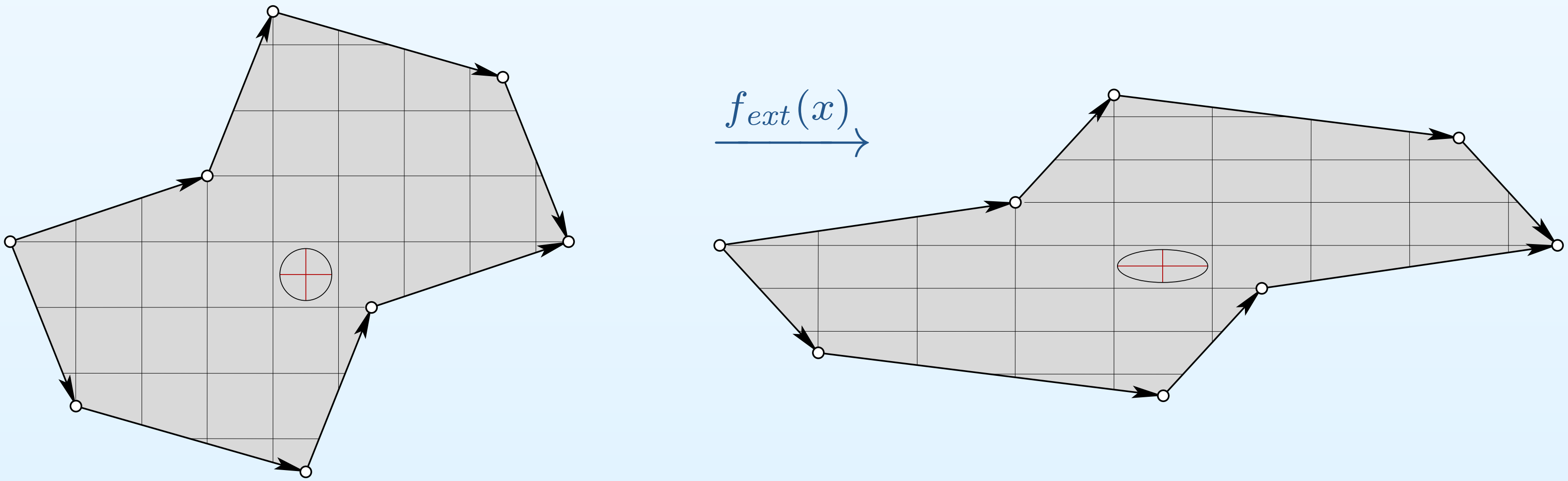
One can choose a pair of holomorphic quadratic differentials q_1 *on* X_1 *and* q_2 *on* X_2*, such that in the corresponding flat coordinates, the map* f_{ext} *acts as an expansion–contraction in respectively horizontal and vertical directions with a constant coefficient* $\sqrt{K(f_{ext})}$*. The foliations correspond to foliations of big (respectively small) demi-axes of ellipses.*



Teichmüller metric and Teichmüller geodesic flow.

Teichmüller metric measures the distance between two complex structures as

$$dist(X_1, X_2) = \frac{1}{2} \log K(f_{ext}),$$

where $f_{ext} : X_1 \to X_2$ is the extremal map. Any holomorphic quadratic differential defines a direction of deformation of the complex structure and a geodesic in the Teichmüller metric. Namely, a holomorphic quadratic differential defines a flat metric. A one-parameter family of maps, which in the flat coordinates are defined by diagonal matrices $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, is a one-parameter family of extremal maps, so it forms a Teichmüller geodesic. According to the definition above we have $dist(X, g_t X) = t$.

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Remarks. The Teichmüller metric is not Riemannian but Finsler: it does not correspond to a quadratic form in the tangent space, but just to a norm which depends continuously on the point of the space of complex structures.

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Taking into consideration a functorial behavior of the vector bundle of holomorphic quadratic differentials, one can see, that it should be identified with a *cotangent* bundle (and not with a *tangent* bundle).

Count of saddle connections and of closed geodesics. Siegel–Veech constants.

Saddle connections

A *saddle connection* is a geodesic segment joining a pair of conical singularities or a conical singularity to itself without any singularities in its interior.

Similar to the torus case regular closed geodesics on flat surface always appear in families; any such family fills a maximal cylinder bounded on each side by a closed saddle connection or by a chain of parallel saddle connections.

Let $N_{sc}(S, L)$ be the number of saddle connections of length at most L on a flat surface S. Let $N_{cg}(S, L)$ be the number of maximal cylinders filled with closed regular geodesics of length at most L on S. It was proved by H. Masur that for any flat surface S both counting functions $N(S, L)$ grow quadratically in L:

$$const_1(S) \le \frac{N(S, L)}{L^2} \le const_2(S)$$

Exact quadratic asymptotics

Theorem (A. Eskin and H. Masur, 2001). *For almost all flat surfaces S of area 1 in any connected $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{L}$ in any stratum of Abelian differentials the counting functions $N_{sc}(S,L)$ and $N_{cg}(S,L)$ have exact quadratic asymptotics*

$$\lim_{L\to\infty}\frac{N_{sc}(S,L)}{\pi L^2}=c_{sc}\qquad \lim_{L\to\infty}\frac{N_{cg}(S,L)}{\pi L^2}=c_{cg}\,.$$

The Siegel–Veech constants c_{sc} and c_{cg} depend only on the suborbifold $\mathcal{L}$.

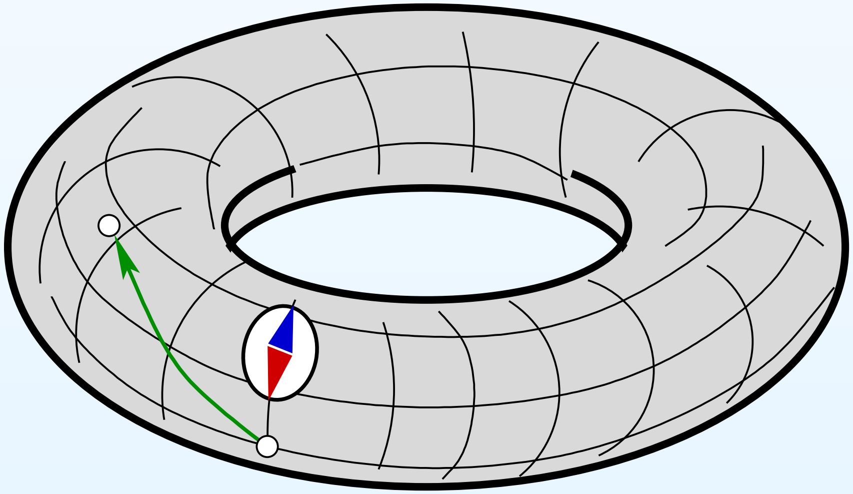
The Magic Wand Theorem of Eskin–Mirzakhani–Mohammadi implies that the above statement is valid for *every* S under extra averaging:

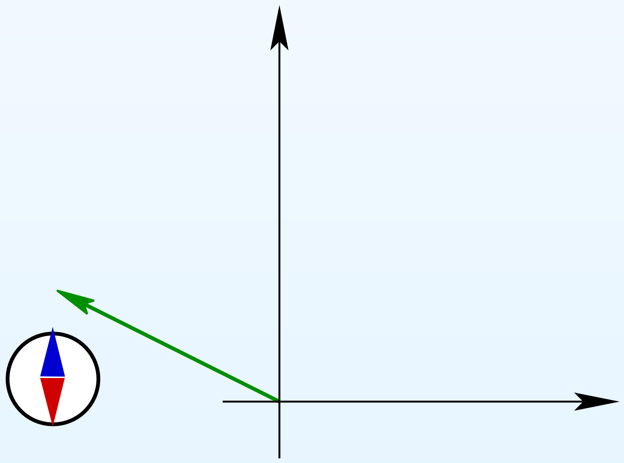
$$\lim_{L\to\infty}\frac{1}{L}\int_0^L N_{sc}(S,e^t)e^{-2t}\,dt=c_{sc};\quad \lim_{L\to\infty}\frac{1}{L}\int_0^L N_{cg}(S,e^t)e^{-2t}\,dt=c_{cg}\,,$$

where the Siegel–Veech constants c_{sc} and c_{cg} depend only on the $\mathrm{SL}(2,\mathbb{R})$-orbit closure $\mathcal{L}=\overline{\mathrm{SL}(2,\mathbb{R})\cdot S}$ of the translation surface S.

Holonomy vector of a saddle connection

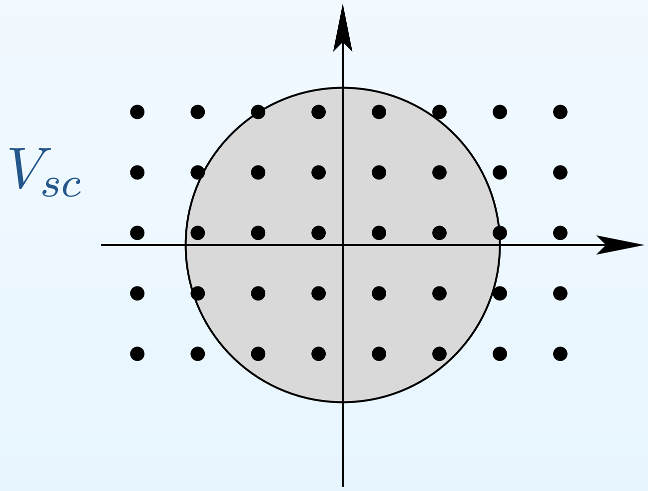
To every saddle connection γ on a flat surface S (or to every closed geodesic, if we want to count closed geodesics) assign a vector $\vec{v}(\gamma)$ in the Euclidean plane $\mathbb{R}^2$ having the length and the direction of γ. In other words, $\vec{v} = \int_\gamma \omega$, where we consider a complex number as a vector in $\mathbb{R}^2 \simeq \mathbb{C}$. We get a discrete set V in $\mathbb{R}^2$.

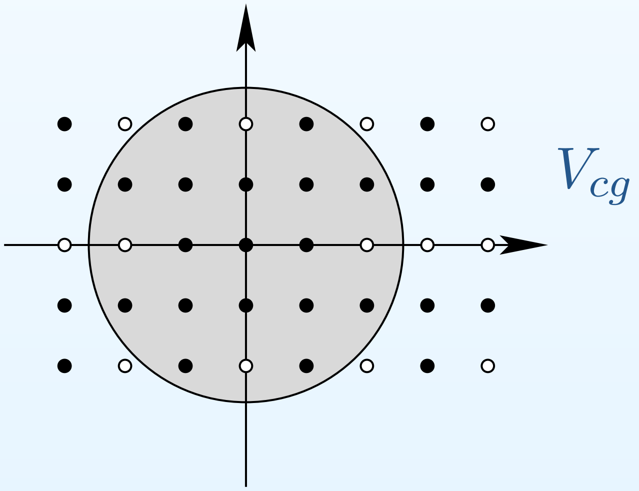




Holonomy sets for saddle connections and for closed geodesics

Mark two points on a torus and consider all geodesic segments joining these two points. They mimic saddle connections. We associate to them a set V_{sc} of holonomy vectors. Consider also all closed geodesics; we associate to them the set V_{cg} of holonomy vectors. To count the number of saddle connections or closed geodesics of length bounded by L is the same as to count the number of points of V_{sc} or V_{cg} which get into a disc of radius L.





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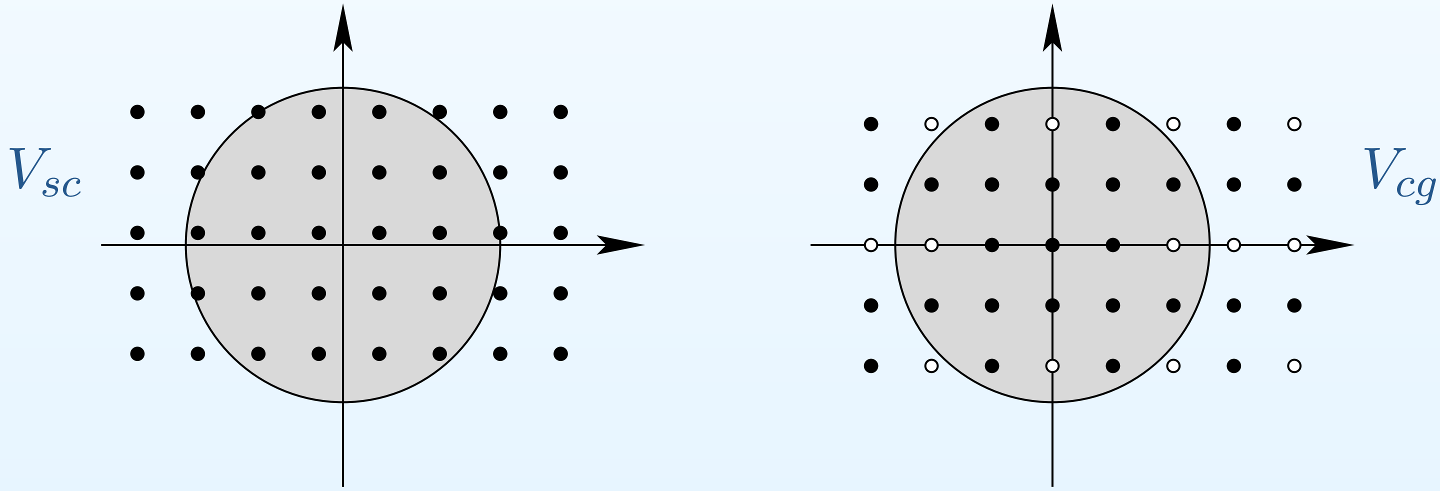

Remark. The discrete sets $V_{sc} \subset \mathbb{R}^2$ and $V_{cg} \subset \mathbb{R}^2$ are transformed equivariantly with respect to the group action:

$$V(gS) = gV(S) \quad \text{for any } g \in \mathrm{GL}(2, \mathbb{R})\,.$$

Siegel–Veech formula

Consider the following operator $f \mapsto \hat{f}$ from functions with compact support on $\mathbb{R}^2$ to functions on the stratum $\mathcal{H}_1(d_1, \dots, d_n)$:

$$\hat{f}(S) := \sum_{\vec{v} \in V(S)} f(\vec{v})$$

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Function $\hat{f}(S)$ generalizes the counting function $N(S, L)$: when $f(x, y)$ is the characteristic function χ_L of the disc of radius L with the center at the origin, $\hat{\chi}_L(S) = N(S, L)$ counts the number of chosen configurations of homologous saddle connections of length at most L on a flat surface S.

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Lemma (W. Veech). *The functional*

$$f \mapsto \int_{\mathcal{H}_1^{comp}(d_1,\dots,d_n)} \hat{f}(S)\, d\nu_1$$

is $\mathrm{SL}(2, \mathbb{R})$*-invariant.*

Siegel–Veech formula

Theorem (W. Veech'98) *For any function $f : \mathbb{R}^2 \to \mathbb{R}$ with compact support*

$$\frac{1}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)}\int_{\mathcal{H}_1(d_1,\dots,d_n)} \hat{f}(S)\, d\nu_1 = C\int_{\mathbb{R}^2} f(x,y)\, dx\, dy\,,$$

where the constant C does not depend on the function f.

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Proof: The only $\mathrm{SL}(2,\mathbb{R})$-invariant functionals are the integral over $\mathbb{R}^2$, the value in the origin, and their linear combinations.

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Proof: The only $\mathrm{SL}(2,\mathbb{R})$-invariant functionals are the integral over $\mathbb{R}^2$, the value in the origin, and their linear combinations.

Theorem (A. Eskin, H. Masur'01) *For almost all flat surfaces S in any connected component of any stratum, the Siegel–Veech constant $c(S)$ in the quadratic asymptotics $N(S,L) \sim c(S)\cdot \pi L^2$, as $L \to \infty$, coincides with the constant C in the Theorem of Veech.*

Remark. The Theorem of Veech allows to count the *average* number of closed geodesics (saddle connections) of length at most L *for any L large or small*. We have an exact equality. Eskin and Masur compute the *asymptotic* number $N(S,L)$ only *when L is large*, but for almost any individual surface S.

Calculation of Siegel–Veech constants: key idea

To compute C it is sufficient to evaluate $\int_{\mathcal{H}_1} \hat{f}(S)\, d\nu_1$ for a single function f. Consider a characteristic function $\chi_\varepsilon(x,y)$ of a disc of a very small radius ε in $\mathbb{R}^2$. Then $\hat{\chi}_\varepsilon(S)$ counts how many ε-short saddle connections (closed geodesics) we can find on a flat surface S. We have

$$\hat{\chi}_\varepsilon(S) = \begin{cases} 0 & \text{for most of the surfaces } S \\ 1 & \text{for } S \in \mathcal{H}_1^{\varepsilon,thick}(d_1,\dots,d_n) \\ >1 & \text{for } S \in \mathcal{H}_1^{\varepsilon,thin}(d_1,\dots,d_n) \end{cases}$$

where $\mathcal{H}_1^{\varepsilon,thin}(d_1,\dots,d_n)$ is the subset of surfaces containing at least two nonhomologous saddle connections of length at most ε. We get

$$\int_{\mathcal{H}_1} \hat{\chi}_\varepsilon(S)\, d\nu_1 = \operatorname{Vol} \mathcal{H}_1^{\varepsilon,thick}(d_1,\dots,d_n) \; + \int_{\mathcal{H}_1^{\varepsilon,thin}} \hat{\chi}_\varepsilon(S)\, d\nu_1 .$$

Calculation of Siegel–Veech constants: key idea

For a characteristic function $\chi_\varepsilon(x,y)$ of a disc of radius ε the Siegel–Veech formula gives us:

$$\frac{1}{\operatorname{Vol}\mathcal{H}_1(d_1,\dots,d_n)}\int_{\mathcal{H}_1}\hat\chi_\varepsilon(S)\,d\nu_1 = C\int_{\mathbb{R}^2}\chi_\varepsilon(x,y)\,dx\,dy = C\cdot\pi\varepsilon^2$$

On the other hand, by definition of $\hat\chi_\varepsilon$, of the thick and the thin parts:

$$\int_{\mathcal{H}_1}\hat\chi_\varepsilon(S)\,d\nu_1 = \operatorname{Vol}\mathcal{H}_1^{\varepsilon,thick}(d_1,\dots,d_n)\;+\;\int_{\mathcal{H}_1^{\varepsilon,thin}}\hat\chi_\varepsilon(S)\,d\nu_1.$$

Theorem (A. Eskin, H. Masur'91)

$$\int_{\mathcal{H}_1^{\varepsilon,thin}}\hat\chi_\varepsilon(S)\,d\nu_1 = o(\varepsilon^2)$$

Calculation of Siegel–Veech constants: the formula

Corollary.

$$\int_{\mathcal{H}_1} \hat{\chi}_\varepsilon(S)\, d\nu_1 = \operatorname{Vol} \mathcal{H}_1^{\varepsilon,thick}(d_1,\dots,d_n) \;+\; o(\varepsilon^2).$$

Applying Siegel–Veech formula we obtain

$$\frac{\operatorname{Vol} \mathcal{H}_1^{\varepsilon}(d_1,\dots,d_n)}{\operatorname{Vol} \mathcal{H}_1(d_1,\dots,d_n)} + o(\varepsilon^2) = C \cdot \pi\varepsilon^2$$

Calculation of Siegel–Veech constants: the formula

Corollary.

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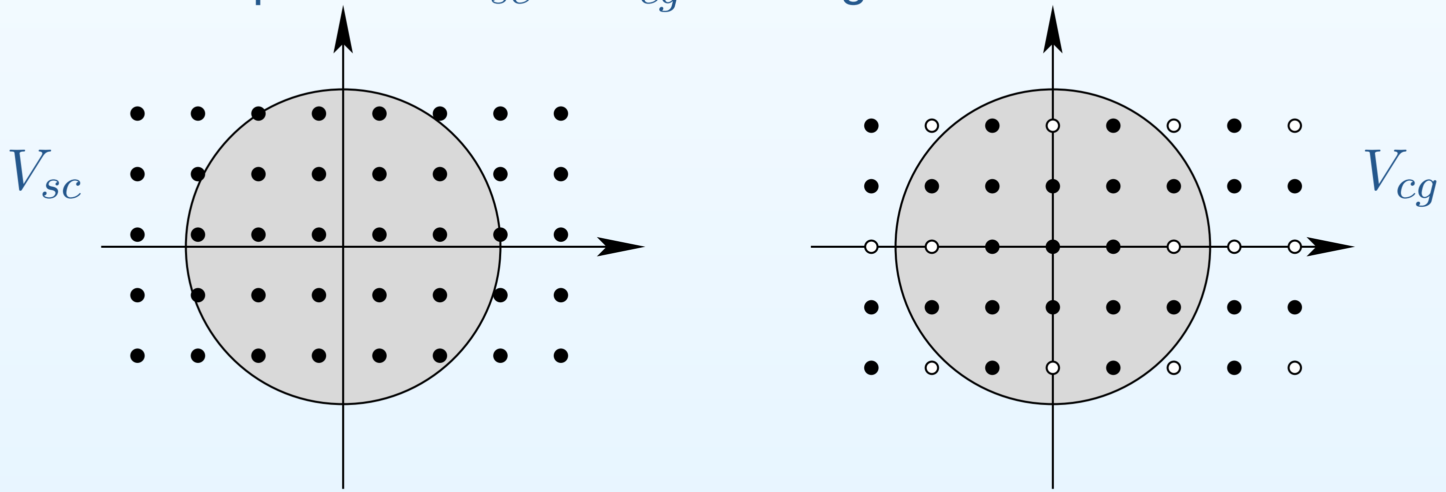
In order to compute the constant C it is sufficient to compute the asymtotics of the volume of the subset $\mathcal{H}_1^{\varepsilon}(d_1,\dots,d_n)$ of surfaces containing a saddle connection of length at most ε, i.e. the volume of a "ε*-thin part*" of $\mathcal{H}_1(d_1,\dots,d_n)$. Then

$$C = \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \frac{\operatorname{Vol}(\text{“}\varepsilon\text{-thin part” of } \mathcal{H}(d_1,\dots,d_n))}{\operatorname{Vol} \mathcal{H}_1(d_1,\dots,d_n)}.$$

Siegel–Veech constants for a flat torus

Siegel–Veech constants for a torus

Mark two points on a torus and consider all geodesic segments joining these two points. They mimic saddle connections. We associate to them a set V_{sc} of holonomy vectors. Consider also all closed oriented geodesics; we associate to them the set V_{cg} of holonomy vectors. To count the number of saddle connections or closed geodesics of length bounded by L is the same as to count the number of points of V_{sc} or V_{cg} which get into a disc of radius L.



If the torus is glued from a unit square, the set V_{sc} is just a shifted lattice, and the set V_{cg} is the set of coprime points in $\mathbb{Z} \oplus \mathbb{Z}$. Thus, the corresponding Siegel–Veech constants should be $c_{sc} = 1$ and $c_{cg} = \frac{6}{\pi^2}$. Let us compute the latter one using our approach.

Volume of the thin part $\mathcal{H}_1^\varepsilon(0)$ of the moduli space of flat tori

Denote by $\mathcal{H}_1(0)$ the space of flat tori of unit area with a chosen direction to the North. Denote by $\mathcal{H}_1^\varepsilon(0)$ the *thin part* of this space, namely the subset of those tori, which have a closed geodesic of length at most ε. Attention to a possible confusion: initially we have decomposed the *thin part* $\mathcal{H}_1^\varepsilon(d_1, \dots, d_n)$ into a disjoint union of a *thick-part-of-the-thin-part* $\mathcal{H}_1^{\varepsilon,thick}(d_1, \dots, d_n)$ and its complement, a *thin-part-of-the-thin-part* $\mathcal{H}_1^{\varepsilon,thin}(d_1, \dots, d_n)$.

Lemma. *The thin part $\mathcal{H}_1^\varepsilon(0)$ of the moduli space of flat tori has Masur–Veech volume $\operatorname{Vol}(\mathcal{H}_1^\varepsilon(0)) = 2\pi\varepsilon^2$.*

Corollary. *The Siegel—Veech constant $c_{cg}(\mathcal{H}(0))$ satisfies:*

$$c_{cg} = \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \cdot \frac{\operatorname{Vol}(\mathcal{H}_1^\varepsilon(0))}{\operatorname{Vol}(\mathcal{H}_1(0))} = \frac{1}{\pi\varepsilon^2} \cdot \frac{2\pi\varepsilon^2}{\pi^2/3} = \frac{6}{\pi^2} = \frac{1}{\zeta(2)}$$

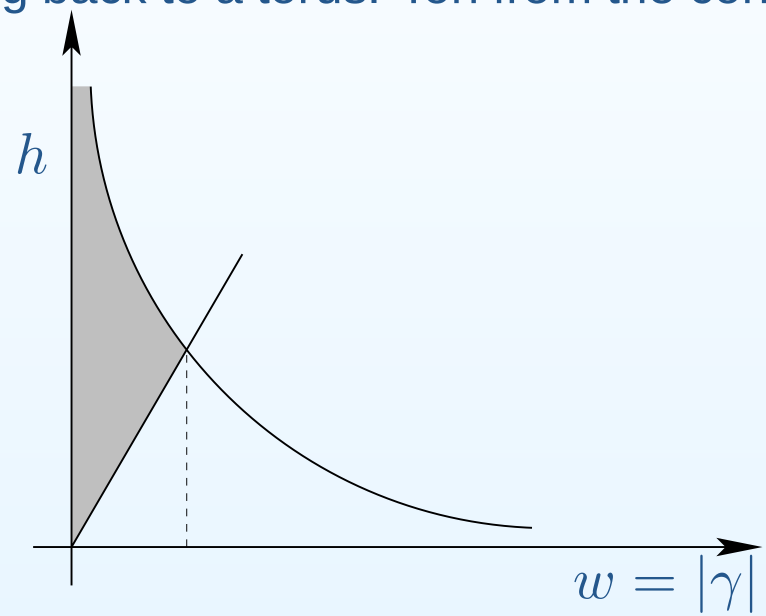
Corollary. *The set of coprime lattice points has density $\dfrac{1}{\zeta(2)}$ in $\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. The set of coprime integer points (m, n) (the ones with $gcd(m, n) = 1$) is exactly the set of holonomy vectors of closed geodesics.

Volume of the thin part $\mathcal{H}_1^\varepsilon(0)$ of the moduli space of flat tori

Proof of the Lemma. We first evaluate the volume $\nu(C(\mathcal{H}_1^\varepsilon(0)))$ of the corresponding cone $C(\mathcal{H}_1^\varepsilon(0)) = \mathcal{H}_{\le 1}^\varepsilon(0))$. Let $|\gamma|$ be the systole in the flat metric, h — the hight of the cylinder obtained by cutting the torus by γ, and t the twist of the cylinder, when gluing back to a torus. Tori from the cone satisfy:

$$\begin{cases} h \cdot |\gamma| \le 1 \\ |\gamma| \le \varepsilon \cdot \sqrt{h \cdot |\gamma|} \\ 0 \le t < |\gamma|\,. \end{cases}$$



Letting $w = |\gamma|$ we get

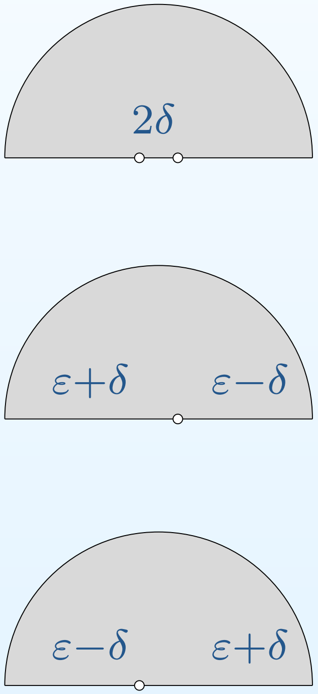
$$\nu(C(\mathcal{H}_1^\varepsilon(0))) = \int_{B(\varepsilon)} d\gamma \int_{|\gamma|/\varepsilon^2}^{1/|\gamma|} dh \int_0^{|\gamma|} dt = 2\pi \int_0^\varepsilon w \left(\frac{1}{w} - \frac{w}{\varepsilon^2}\right) w\, dw = \frac{\pi\varepsilon^2}{2}.$$

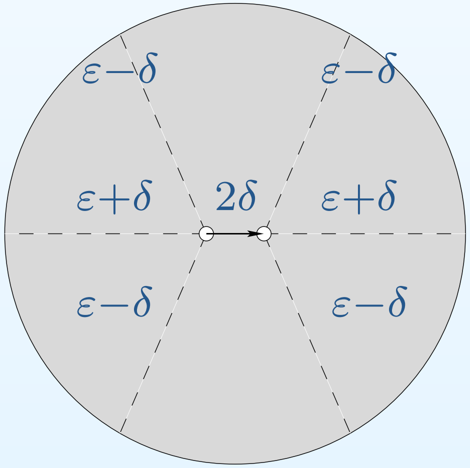
It remains to recall that $\nu(C(\mathcal{H}_1^\varepsilon(0))) = \dim_{\mathbb{R}} \mathcal{H}(0) \cdot \operatorname{Vol}(\mathcal{H}_1^\varepsilon(0))$ where $\dim_{\mathbb{R}} \mathcal{H}(0) = 4$. $\square$

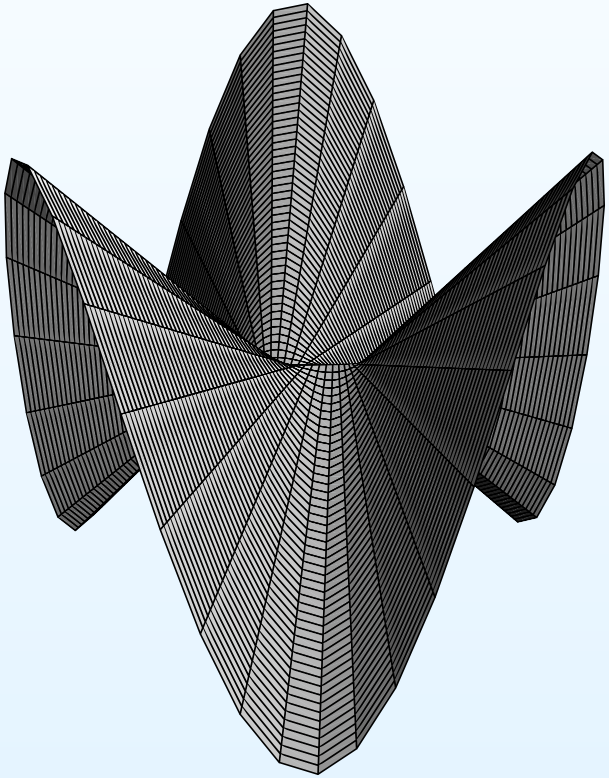
Breaking up a zero into two

Breaking up a double zero into two simple ones

Cut an ε-neighborhood of the double zero out of the surface. Decompose it into six metric half-disks of radius ε. Now change identifications of diameters of these half-discs as indicated and paste the result into the surface.







Volume of thin part of $\mathcal{H}_1(1,1)$

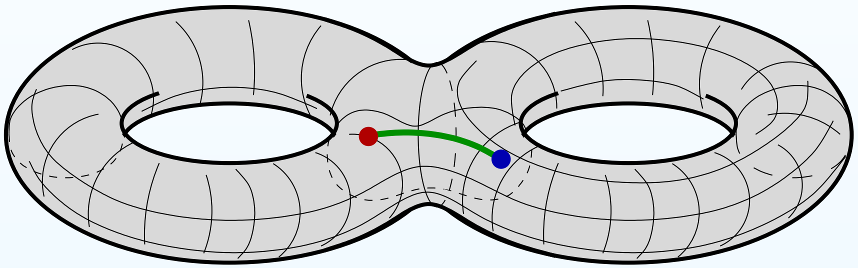
We want to compute the measure of the subset of surfaces having a single short saddle connection joining two simple zeroes. There is a canonical way to shrink the saddle connection on $S \in \mathcal{H}_1^{\varepsilon,thick}(1,1)$ coalescing two zeroes into one. This provides us with an (almost) fiber bundle

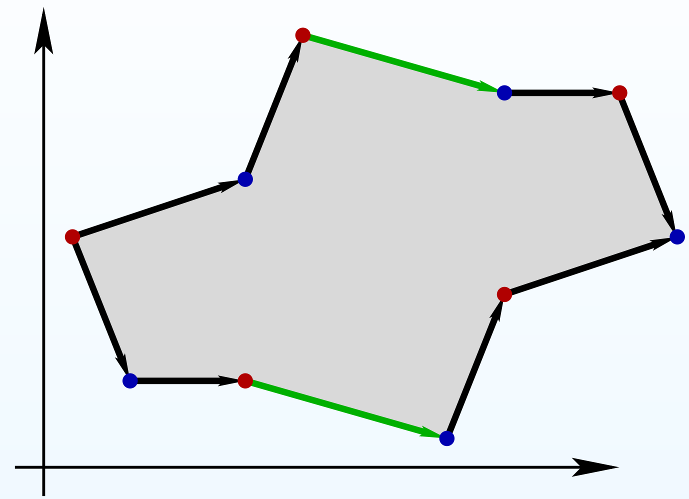
$$\begin{array}{c} \mathcal{H}_1^{\varepsilon,thick}(1,1) \\ \Big\downarrow \tilde{D}^2_\varepsilon \\ \mathcal{H}_1(2) \end{array}$$

where $\tilde{D}^2_\varepsilon$ is a ramified cover of order 3 over a standard metric disc of radius ε. Moreover, the measure on $\mathcal{H}_1^{\varepsilon,thick}(1,1)$ disintegrates into a product of the standard measure on $\tilde{D}^2_\varepsilon$ and the natural measure on $\mathcal{H}_1(2)$ which implies:

$$\mathrm{Vol}(\text{“}\varepsilon\text{-thin part” of } \mathcal{H}(1,1)) \sim 3 \cdot \pi\varepsilon^2 \cdot \mathrm{Vol}\,\mathcal{H}_1(2)\,.$$

Siegel–Veech constant for saddle connections on $\mathcal{H}_1(1,1)$





Plugging the resulting expression for $\mathrm{Vol}(\text{“}\varepsilon\text{-thin part” of } \mathcal{H}(1,1))$ into the formula for the Siegel–Veech constant we get

$$c_{sc}(\mathcal{H}(1,1)) = \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \frac{\mathrm{Vol}(\text{“}\varepsilon\text{-thin part” of } \mathcal{H}(1,1))}{\mathrm{Vol}\,\mathcal{H}_1(1,1)} = \frac{3\,\mathrm{Vol}\,\mathcal{H}_1(2)}{\mathrm{Vol}\,\mathcal{H}_1(1,1)} = 3\,\frac{\frac{\pi^4}{120}}{\frac{\pi^4}{135}} = \frac{27}{8}\,.$$

Phenomenon of higher multiplicities

Phenomenon of multiple saddle connections

Consider some saddle connection $\gamma_1 = [P_1 P_2]$ with an endpoint at P_1. Memorize its direction, say, let it be the North-West direction. Let us launch a geodesic from the same starting point P_1 in one of the remaining $k-1$ North-West directions. Let us study how big is the chance to hit P_2 ones again, and how big is the chance to hit it after passing the same distance as before.

Theorem (A. Eskin, H. Masur, A. Zorich'03). *For almost any flat surface S in any stratum and for any pair P_1, P_2 of conical singularities on S the function $N_2(S,L)$ counting the number of pairs of parallel saddle connections of the same length joining P_1 to P_2 also has exact quadratic asymptotics*

$$\lim_{L\to\infty} \frac{N_2(S,L)}{\pi L^2} = c_2 > 0.$$

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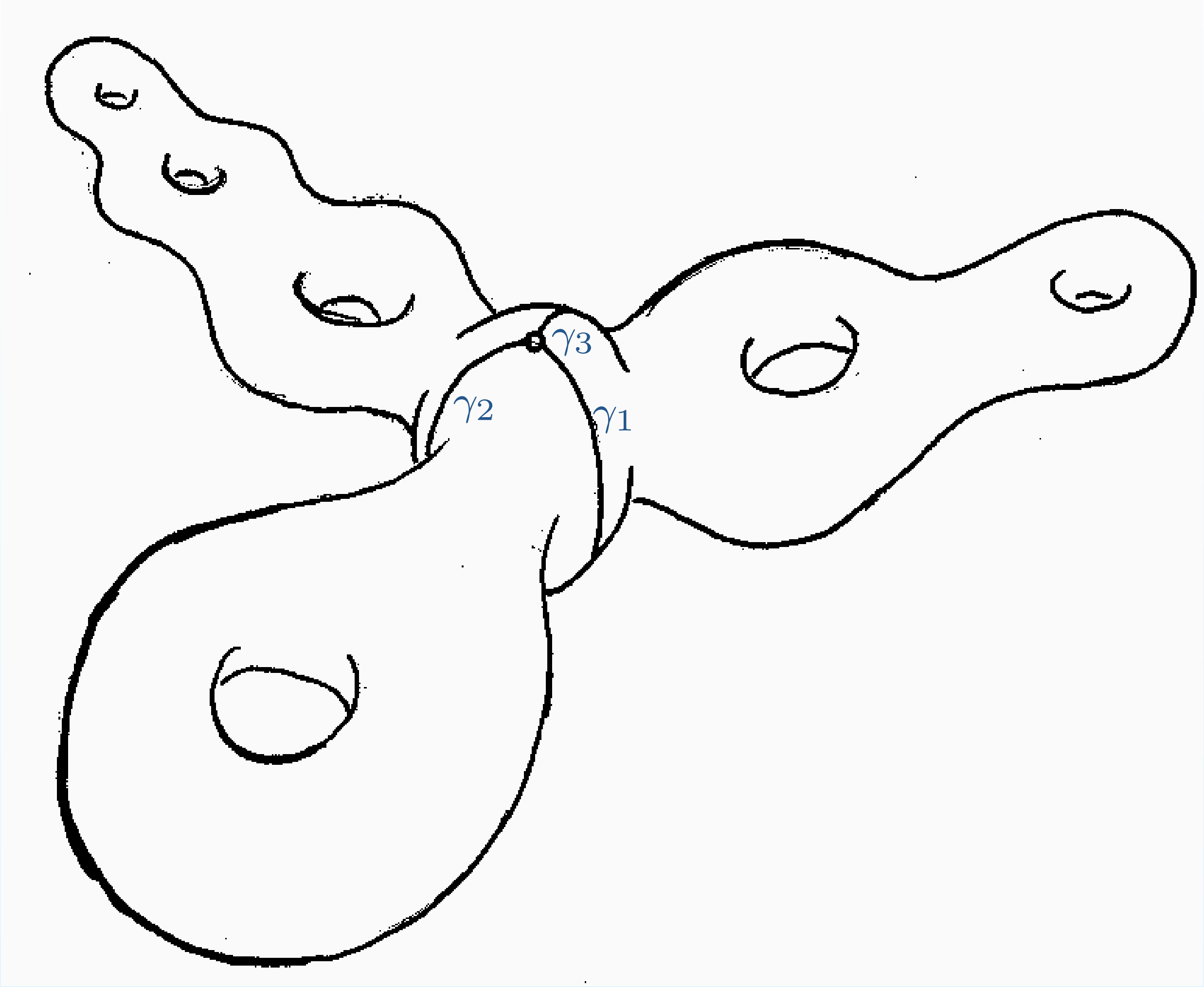
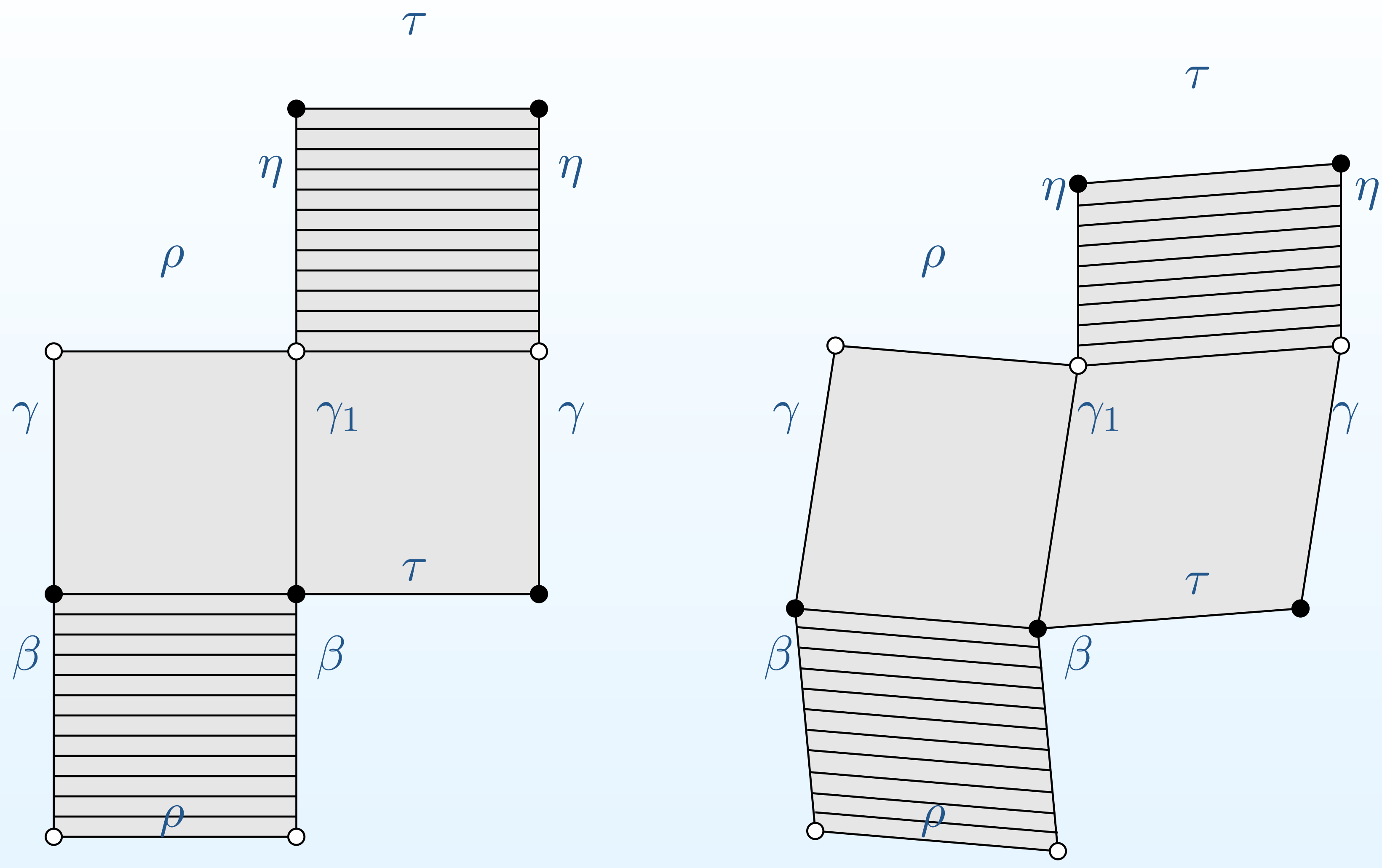
However, for almost all flat surfaces S in any stratum one cannot find neither a single pair of parallel saddle connections on S of different length, nor a single pair of parallel saddle connections joining different pairs of singularities.

Saddle connections joining distinct zeroes



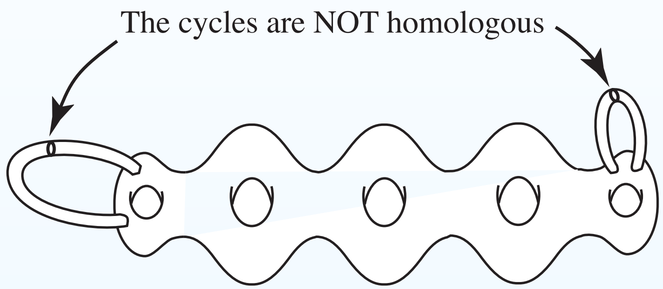
Multiple homologous saddle connections, topological picture.

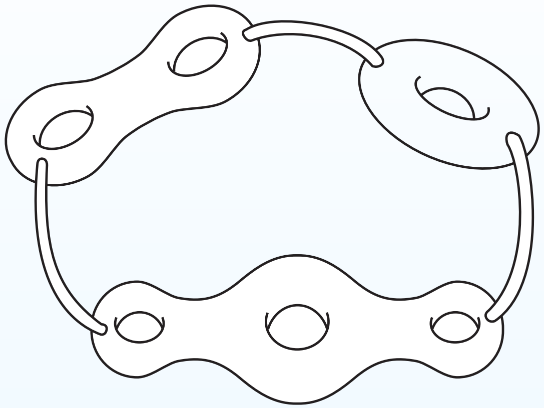
Saddle connections joining distinct zeroes



Saddle connections γ and γ_1 are homologous. They stay parallel and isometric, $|\gamma_1| = |\gamma|$, under any small deformation of the flat surface.

Typical and nontypical degenerations





Theorem (H. Masur, J. Smillie'91) *The set of surfaces as on the right such that the waist curve of the cylinder is shorter than* ε *has measure* $O(\varepsilon^2)$ *in* $\mathcal{H}_1(d_1, \ldots, d_n)$ *no matter what is the number of components.*
The set of surfaces as on the left such that the waist curve of the cylinder is shorter than ε *has measure* $O(\varepsilon^4)$.

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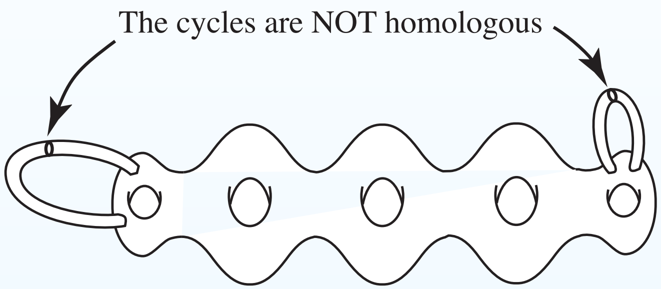

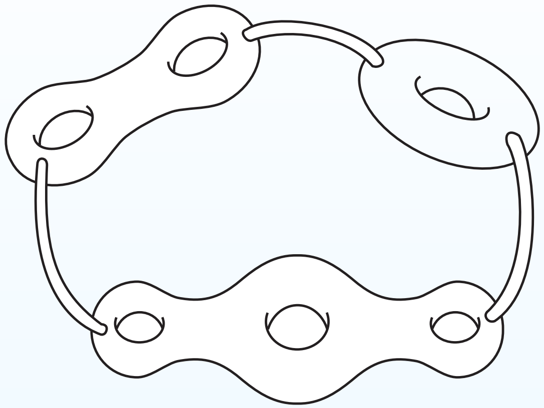

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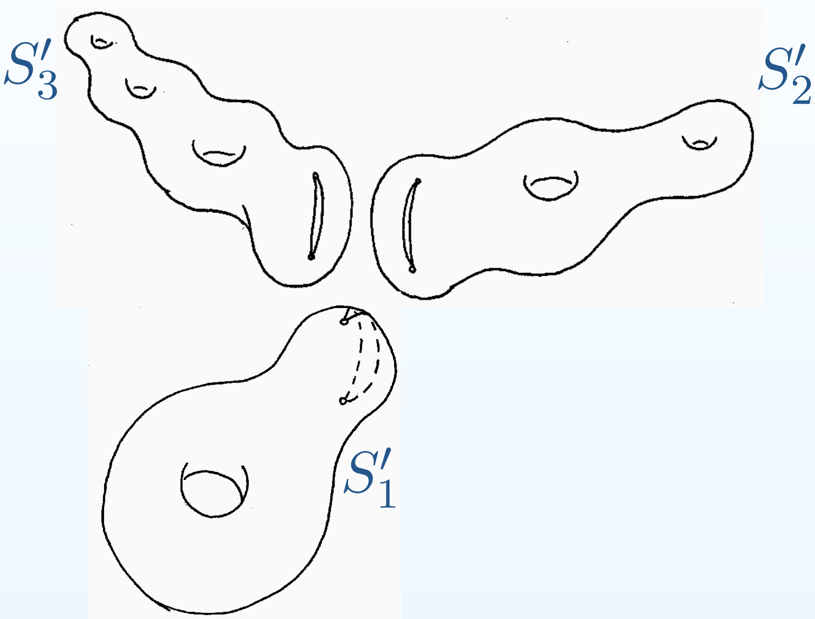
The set of surfaces as on the left such that the waist curve of the cylinder is shorter than ε has measure $O(\varepsilon^4)$.

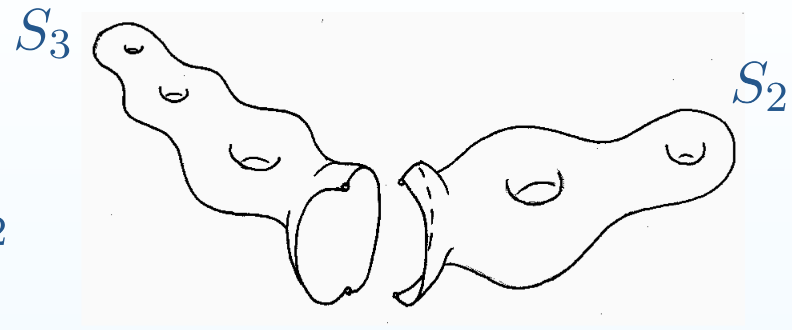
A similar statement is true for short saddle connections. In our language:

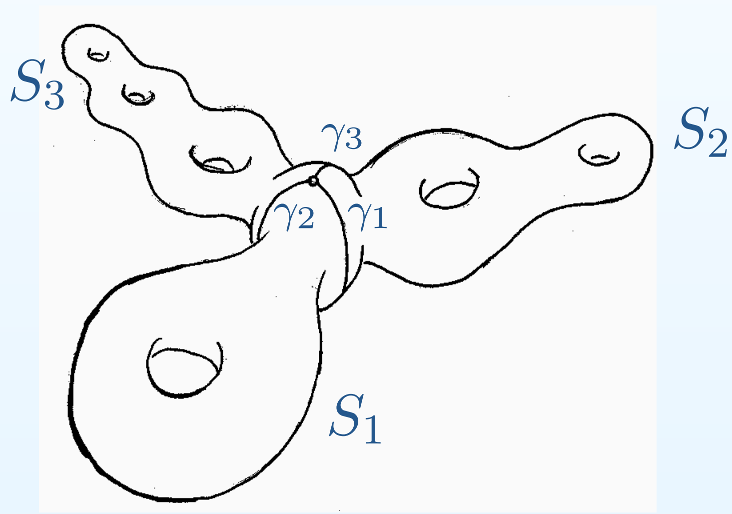
$$\operatorname{Vol} \mathcal{H}_1^{\varepsilon,thin}(d_1, \dots, d_n) = O(\varepsilon^4) \,.$$

More artistic picture of a generic degeneration

Warning: invisible components of stable curves







Contracting slits we have an illusion of getting a “forbidden stable curve” in Deligne–Mumford compactification: it has a triple node. Actually, the underlying complex curve develops an extra $\mathbb{C}\mathrm{P}^1$ through which the three components are attached. This hardly visible component has zero limiting flat area. An adequate compactification is recently constructed by Bainbridge–Chen–Gendron–Grushevsky–Möller.

Some recent results

Counting ignoring multiplicities

Working with translation surfaces (holomorphic forms) in $\mathcal{H}(m_1, \ldots, m_n)$ one usually labels all conical singularities $P_1, \ldots, P_n$. Fix any two of them, P_i and P_j. Let us count saddle connections joining P_i to P_j neglecting multiplicities (i.e., let us count saddle connections looking only at their holonomy vectors in $\mathbb{R}^2$). The corresponding Siegel–Veech constant $c_{i,j}^{hom}$ is the sum of all Siegel–Veech constants corresponding to all possible configurations of homologous saddle connections joining P_i to P_j.

Theorem (D. Chen, M. Möller, A. Sauvaget, D. Zagier, 2020). *For any nonhyperelliptic component of any stratum $\mathcal{H}(m_1, \ldots, m_n)$ of Abelian differentials one has* $c_{i,j}^{hom} = (m_i + 1)(m_j + 1)$.

The formula has the following (somehow misleading) heuristic interpretation: the cone angle $2\pi(m_i + 1)$ at the conical point P_i is $(m_i + 1)$ times larger than at a regular point. So there are $(m_i + 1)$ times more saddle connections getting out of P_i than from a regular point. Multiplying, $(m_i + 1)$ by $(m_j + 1)$ we get the answer.

There are yet no analogous formulae valid for quadratic differentials!

Area Siegel—Veech constant

Closed regular geodesics on flat surfaces appear in families of parallel closed geodesics sharing the same length. Every such family fills a *maximal cylinder* having conical points on each of the boundary components. We have seen that sometimes we might get a *configuration* $\mathcal{C}$ of several cylinders, with homologous waste curves (sharing the same length and direction).

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Denote by $N_{area}(S, L)$ the sum of areas of all cylinders spanned by geodesics of length at most L on a translation surface S of area 1.

Theorem [W. Veech; Ya. Vorobets] *For every* $\mathrm{SL}(2, \mathbb{R})$*-invariant finite ergodic measure the following ratio is constant (i.e. does not depend on the value of a positive parameter* L*):*

$$\frac{1}{\pi L^2} \int N_{area}(S, L)\, d\nu_1 = c_{area}(d\nu_1)$$

The constant c_{area} is called the *area Siegel–Veech constant*.

Large genus asymptotics

The result below (in a slightly weaker form) was conjectured by A. Eskin and A. Zorich about 2003. The conjecture was proved in 2020 by D. Chen, M. Möller, A. Sauvaget, D. Zagier, and independently in 2019 by A. Aggarwal (in a slightly weaker form by different methods).

Theorem. *For any nonhyperelliptic component of any stratum* $\mathcal{H}(m_1,\ldots,m_n)$ *of Abelian differentials one has*

$$c_{area} = \frac{1}{2} - \frac{1}{2\sum_{i=1}^{n}(m_i+1)} + O(1/g^2) \text{ as } g \to +\infty\,,$$

where the implied constants are independent of the partition $m_1+\cdots+m_n = 2g-2$ *and of* g.

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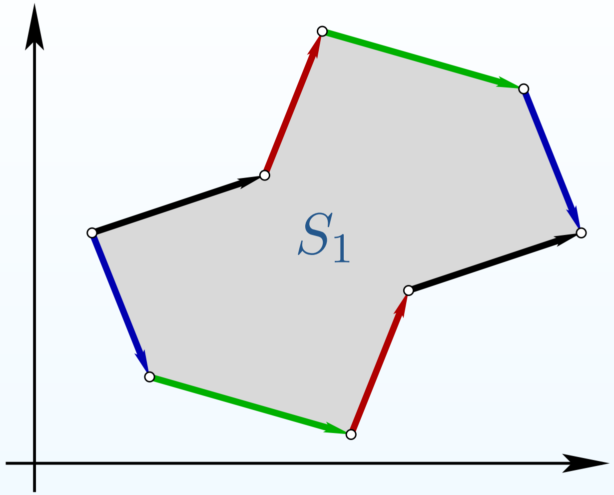
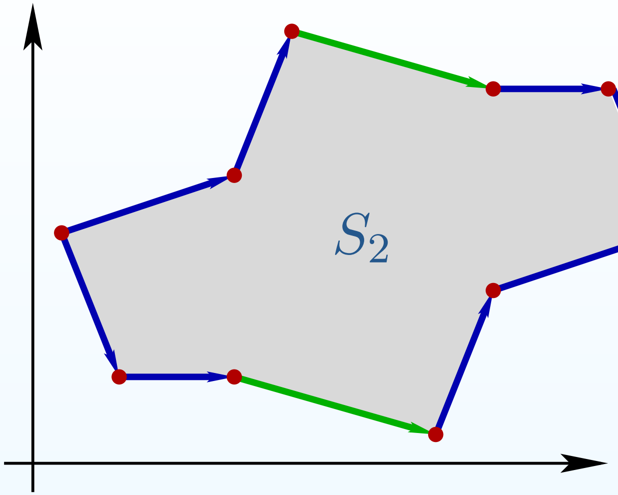
Combining the theorem with further results of D. Chen, M. Möller, A. Sauvaget, D. Zagier, and of A. Aggarwal on large genus asymptotics of Masur–Veech volumes (confirming another conjecture of A. Eskin and A. Zorich) one gets

Theorem (A. Zorich'20). *The relative contribution of all configurations of saddle connections of multiplicity 2 and more to c_{area} and to $c_{i,j}^{hom}$ tends to 0 uniformly in partitions $m_1 + \dots + m_n = 2g-2$ and in g as $g \to +\infty$.*

Hints for the exercise

Hyperelliptic involution and Weierstrass points





● Verify that the surface S_1, obtained by identifying pairs of corresponding sides of the first polygon (respectively S_2 — of the second polygon) by parallel translations, have genus 2. To which strata belong S_1 and S_2?

● It is known that every Riemann surface of genus 2 is *hyperelliptic*, i.e. it admits a holomorphic involution τ such that the quotient over the involution is $\mathbb{C}\mathrm{P}^1$. Describe the hyperelliptic involutions for the surfaces S_1 and S_2.

● Fixed points of a hyperelliptic involution are called *Weierstrass points*. It follows from the Riemann–Hurwitz formula (which is a nice and very simple fact) that there are $2g+2$ Weierstrass points. Find all Weierstrass points for the surfaces S_1 and S_2.