2021-11-15 Kähler geometry

M Fano manifold

$$\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \in 2\pi c_1(M)$$

$$\mathrm{Ric}(\omega) - \omega = i\partial\bar{\partial} F_\omega \qquad F = F_\omega$$ omitted.

$$F \in C^\infty(M)$$

Aubin's continuity method

$$*_t \quad \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-t\varphi + F}$$

sol for $t=1 \Longrightarrow$ KE metric

$S = \{ t \in [0,1] \mid *_t$ has a solution $\}$.

$$\left\{ \begin{array}{l} \cdot\ S \text{ is non-empty} \rightsquigarrow 0 \in S,\ \text{Yau's solution} \\ \cdot\ S \text{ is open} \rightsquigarrow \text{implicit fn thm (Delanoë)} \\ \cdot\ S \text{ is closed} \rightsquigarrow \boxed{\text{difficult.}} \end{array} \right.$$

$\Longrightarrow S = [0,1] \Longrightarrow S \ni 1$

Closedness

When $\underset{\in S}{t_i} \to t_\infty$, $\underline{t_\infty \in S\ ?}$

②

$\mathrm{Ric}(g_{t_i}) > t g_i$

$\exists_t \Leftrightarrow \mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega_0 > t\omega_t$

Cheeger-Colding theory

$(M, g_{t_i}) \longrightarrow M_\infty$ Gromov-Hausdorff

Demerit. • It is difficult to understand the singularity

• It is unclear how to use K-stability.

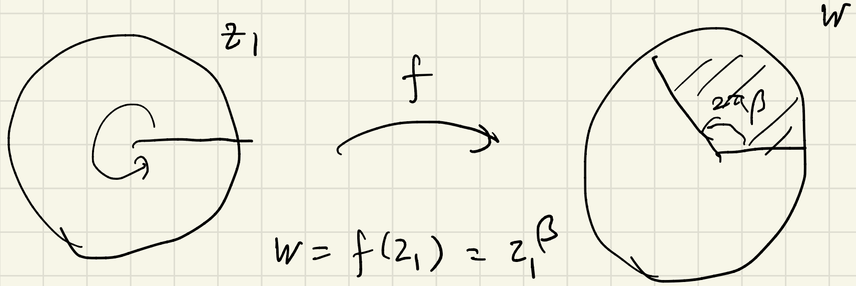
Another continuity method using cone angle KE metrics. $2\pi\beta \leftarrow$ cone angle

M (compact) Kähler manifold

D smooth divisor $D = \{z_1 = 0\}$ locally

local coord $(z_1, \cdots, z_m)$

local model of a Kähler metric with cone angle $2\pi\beta$



$$w = f(z_1) = z_1^{\beta}$$

$$f^*\left(i\,dw \wedge d\bar{w} + i\,dz_2 \wedge d\bar{z}_2 + \cdots + i\,dz_m \wedge d\bar{z}_m \right)$$

$$= \beta^2 |z_1|^{2(\beta-1)}\, i\,dz_1 \wedge d\bar{z}_1 + i\,dz_2 \wedge d\bar{z}_2 + \cdots$$

$$+ i\,dz_m \wedge d\bar{z}_m$$

<u>Def</u> A Kähler form ω with cone angle $2\pi\beta$ along a divisor D is when ω is uniformly equivalent to

$$\omega_{cone} = \beta^2 \frac{i\,dz_1 \wedge d\bar{z}_1}{|z_1|^{2-2\beta}} + \sum_{j=2}^{m} i\,dz_j \wedge d\bar{z}_j$$

where $D \cap U = \{ z_1 = 0 \}$,

i.e. $\exists\, c_1, c_2 > 0$ constant s.t.

$$c_1\, \omega_{cone} \le \omega \le c_2\, \omega_{cone}.$$

<u>Rm</u> $\beta = 1 \Longrightarrow \omega$ is smooth.

<u>Def</u> Ricci current $\mathrm{Ric}(\omega)$

$$\mathrm{Ric}(\omega) = -i\partial\bar\partial \log \omega^m$$ distribution derivative

<u>Lemma</u> Write $\mathrm{Ric}(\omega)$ to be the Ricci form on $M \setminus D$. Then

$$\mathrm{Ric}(\omega) = \mathrm{Ric}(\omega) + 2\pi(1-\beta)[D].$$

as a current.

∵ Poincaré-Lelong equation

$$\frac{i}{2\pi}\partial\bar\partial \log|z_1|^2 = [D]$$

$\boxed{\text{Next we assume } M \text{ Fano.}}$

Suppose, $[D] = \lambda c_1(M)$

D smooth

(In algebraic geometry it is unknown whether we can take $\lambda = 1$)

<u>Def</u> A Kähler form ω in $2\pi c_1(M)$ with cone angle $2\pi\beta$ along D is a Kähler-Einstein metric if

$$\mathrm{Ric}(\omega) = \mu\omega + 2\pi(1-\beta)[D] \qquad *_\beta$$

(A version of twisted KE metric)

Remark $\mu = 1 - (1-\beta)\lambda$.

☺ $\underset{\Uparrow}{\mathrm{Ric}(\omega)} = \mu\underset{\Uparrow}{\omega} + 2\pi(1-\beta)[D]$.

$2\pi c_1(M)$ $\qquad 2\pi\left(\mu c_1(M) + (1-\beta)\lambda c_1(M)\right)$

$$\therefore\ 1 = \mu + (1-\beta)\lambda.$$ ☺

$E = \{\beta \in [0,1] \mid *_\beta$ has a solution $\}$.

(i) $E \neq \emptyset$ Brendle, Jeffres-Mazzeo-Rubinstein

(ii) E is open — CDS ⇝ can be reduced orbifold

(iii) E is closed. if M is K-polystable. Donaldson & Song, CT. Wang.

ω_0 smooth Kähler form $[\omega_0] = 2\pi c_1(M)$.

$\mathcal{H} = \{\varphi \in C^\infty(M) \mid \omega_0 + i\partial\bar\partial\varphi > 0\}$.

$\beta \in [0,1]$

$\hat{\mathcal{H}}_\beta = \{\varphi \in C^\infty(M \setminus D) \mid \omega_0 + i\partial\bar\partial\varphi$ Kähler form of cone angle $2\pi\beta$ along $D\}$.

$K_M^{-\lambda}$

Example $\varepsilon > 0$

$\varphi \in \mathcal{H} \Rightarrow \varphi + \varepsilon|s|_h^{2\beta} \in \hat{\mathcal{H}}_\beta$ of $[D]$

where $s \in H^0(M, [D])$, $(s) = D$, h metric

(6)

Lemma For $\varphi_\beta \in \hat{\mathcal{H}}_\beta$ we put $\omega_\beta = \omega_0 + i\partial\bar\partial\varphi_\beta$.

If we define h_β by

$$\mathrm{Ric}(\omega_\beta) = \mu\,\omega_\beta + (1-\beta)2\pi[D] + \sqrt{-1}\partial\bar\partial h_\beta.$$

then

$$h_\beta = h_{\omega_0} - (1-\beta)\log\|S\|^2_{h_0^\lambda} - \log\frac{\omega_\beta^m}{\omega_0^m} - \mu\varphi_\beta$$

h_0 metric $K_M^{-1} \rightsquigarrow h_0^\lambda$ metric for $K_M^{-\lambda} = [D]$.

where $c_1(h_0) = \omega_0$, $\mathrm{Ric}(\omega_0) - \omega_0 = i\partial\bar\partial h_{\omega_0}$

$\because$

$i\partial\bar\partial(\mathrm{RHS})$

$= \cancel{\mathrm{Ric}(\omega_0)} - \omega_0 + (1-\beta)\lambda\omega_0 - (1-\beta)[D]$

$\quad + \mathrm{Ric}(\omega_\beta) - \cancel{\mathrm{Ric}(\omega_0)} - \mu(\omega_\beta - \omega_0)$

$-1 + (1-\beta)\lambda + \mu = 0$

$= \mathrm{Ric}(\omega_\beta) - (1-\beta)[D] - \mu\,\omega_\beta$

$= \sqrt{-1}\partial\bar\partial h_\beta.$ $\quad\blacksquare$

Cor $\varphi_\beta \in \hat{\mathcal{H}}_\beta$ is a solution to $(*_\beta)$

$$\Longleftrightarrow \quad \omega_{\varphi_\beta}^m = e^{-\mu\varphi_\beta + h_{\omega_0}} \frac{\omega_0^m}{\|S\|_h^{2(1-\beta)}} \qquad (*_\beta)$$