Integrable origin and applications of Gibbsian line ensembles

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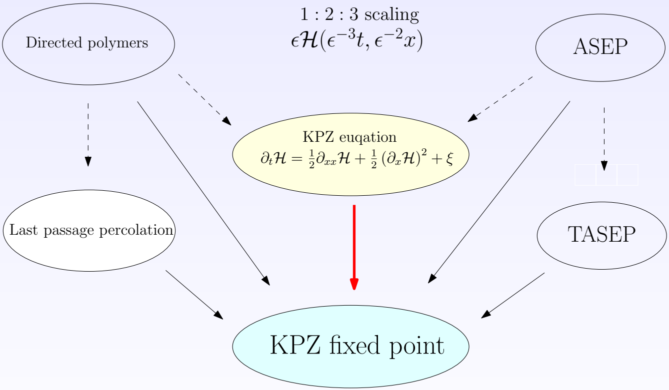
Outline

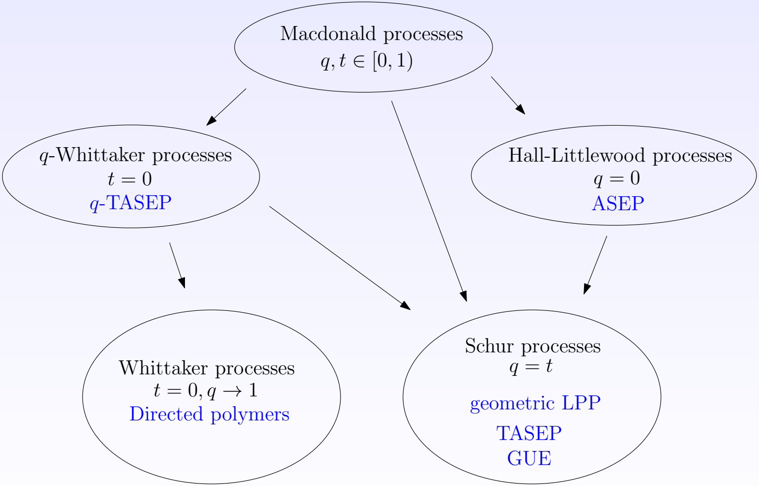
- KPZ universality class and KPZ line ensemble
- Main results
- Gibbs property
- Application of Gibbs property

KPZ universality class and KPZ line ensemble

KPZ Universality



Algebraic hierarchy



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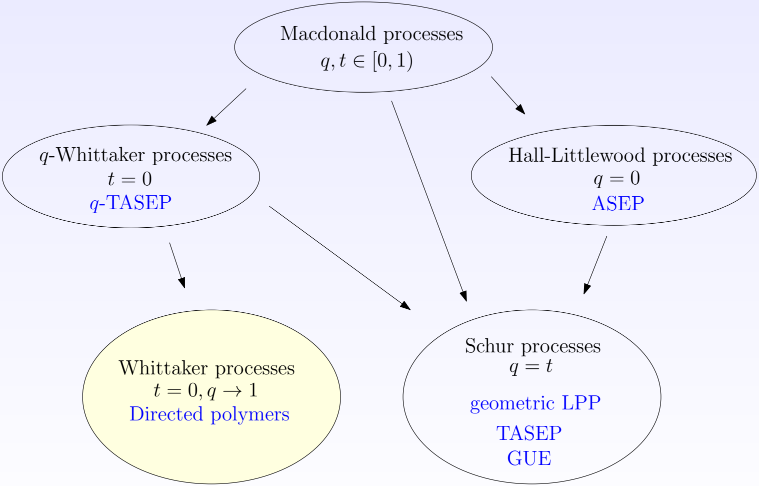
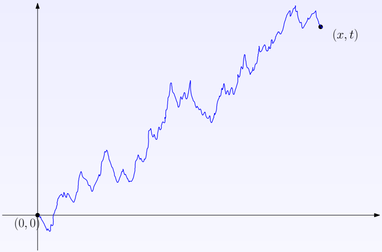


SHE as continuum polymers

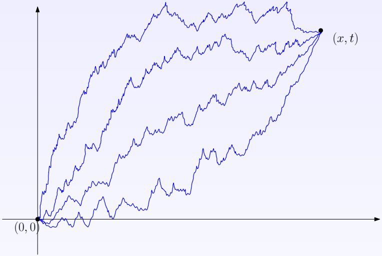


$$\partial_t \mathcal{Z}_1 = \frac{1}{2}\partial_{xx}\mathcal{Z}_1 + \xi \cdot \mathcal{Z}_1$$

$$\mathcal{Z}_1(0, x) = \delta_{x=0}$$

$$\mathcal{Z}_1(t, x) = \mathbb{E}\left[e^{\int_0^t \xi(s, B_1(s))dt}\right]$$

SHE as continuum polymers



$$\mathcal{Z}_k(t,x) = \mathbb{E}\left[e^{\sum_{i=1}^{k} \int_0^t \xi(s, B_i(s))\, dt} \right]$$

From SHE to KPZ

SHE:

$$\partial_t \mathcal{Z}_1 = \frac{1}{2}\partial_{xx}\mathcal{Z}_1 + \xi \cdot \mathcal{Z}_1$$
$$\mathcal{Z}_1(0,x) = \delta_{x=0}$$

KPZ equation:

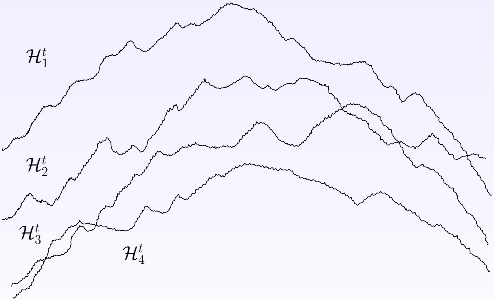
$$\partial_t \mathcal{H}_1 = \frac{1}{2}\partial_{xx}\mathcal{H}_1 + \frac{1}{2}(\partial_x \mathcal{H}_1)^2 + \xi$$
$$\mathcal{H}_1(0,x) = \log \delta_{x=0}$$

Hopf-Cole solution:

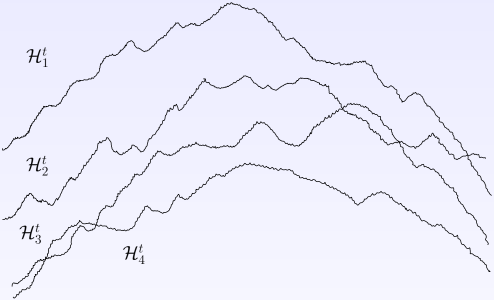
$$\mathcal{H}_1(t,x) = \log \mathcal{Z}_1(t,x),$$

KPZ$_t$ line ensemble

$$\mathcal{H}^t_k(x) = \log \mathcal{Z}_k(t,x) - \log \mathcal{Z}_{k-1}(t,x)$$



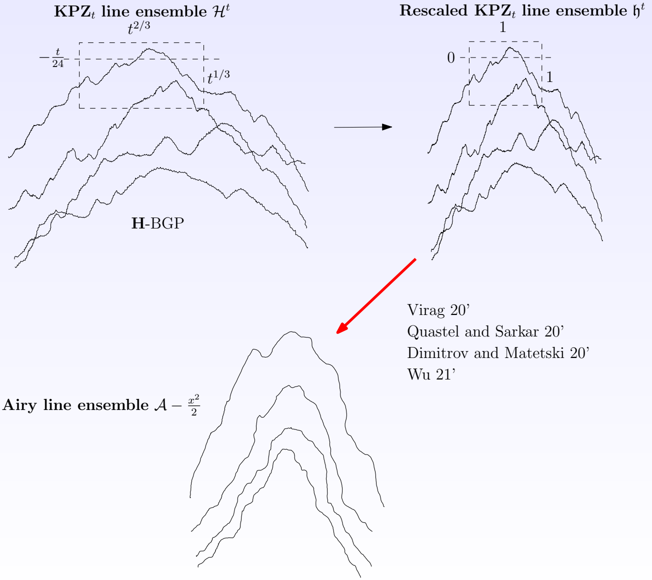
KPZ$_t$ line ensemble, Corwin-Hammond'16.



Fix $t > 0$, KPZ$_t$ line ensemble $\mathcal{H}^t$ enjoys

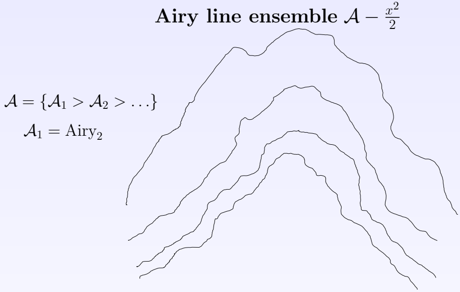
- $\mathcal{H}_1^t =$ time t solution to KPZ equation with narrow wedge initial data.
- **Gibbs property**.

Big picture: convergence to Airy line ensemble.



Virag 20'
Quastel and Sarkar 20'
Dimitrov and Matetski 20'
Wu 21'

Airy line ensemble.



- Random matrix theory, Dyson Brownian motion
- Last passage percolation, Polymer free energy
- TASEP, ASEP
- Multi-layer PNG model
- Brownian watermelon

Main results

Tightness of the KPZ line ensemble

Theorem (Wu 21)

Scaled KPZ line ensemble $\mathfrak{h}^t$ is tight in the locally uniform topology as t goes to infinity.

Tail estimates

Theorem (Wu 21)

For any $a, b \in \mathbb{R}$*, take* $k \in \mathbb{N}$*, there exists* $C(k)$ *such that*

- *Uniform inf bound*

$$\mathbb{P}_{\mathfrak{h}}\left[\inf_{x\in[a,b]}\left(\mathfrak{h}_k^t(x)+\frac{x^2}{2}\right)\leq -R\right]\leq C(k)e^{-C(k)^{-1}R^{3/2}}$$

- *Uniform sup bound*

$$\mathbb{P}_{\mathfrak{h}}\left[\sup_{x\in[a,b]}\left(\mathfrak{h}_k^t(x)+\frac{x^2}{2}\right)\geq R\right]\leq C(k)e^{-C(k)^{-1}R^{3/2}}$$

Local fluctuation estimate

Theorem (Wu 21)

There exist $C(k) > 0$ *such that, for all* $\epsilon \in (0,1]$ *and* $K > 0$, *we have*

$$\mathbb{P}_{\mathfrak{H}}\left(\sup_{\substack{-1 \le x < y \le 1 \\ |x-y| \le \epsilon}} |\mathfrak{H}_k^t(x) - \mathfrak{H}_k^t(y)| \ge K\sqrt{\epsilon} \right) \le C(k) e^{-C(k)^{-1} K^{3/2}}$$

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We know for Brownian Bridges

$$\mathbb{P}_{BB}\left(\sup_{\substack{-1\le x<y\le 1\\ |x-y|\le\epsilon}} |B(x) - B(y)| \geq K\sqrt{\epsilon}\right) \leq e^{-K^2}$$

Brownian comparison for KPZ line ensemble

Let $\delta > 0$, take an arbitrary event A such that $\mathbb{P}_B[A] = \delta$.

Brownian comparison for KPZ line ensemble

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Question: How is $\mathbb{P}_{\mathfrak{h}^t}[A]$ comparable to δ?

Brownian comparison for KPZ line ensemble

Let $\delta > 0$, take an arbitrary event A such that $\mathbb{P}_B[A] = \delta$.

Theorem (Wu 21)

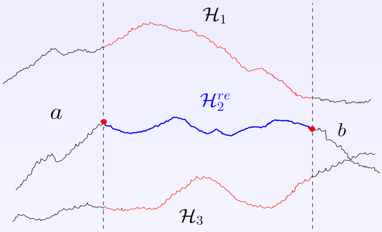
$$\mathbb{P}_{\mathfrak{h}^t}[A] \le \delta \cdot \exp\left\{\left(\log \delta^{-1})^{5/6}\right) O(1)\right\} = \delta^{1-o(1)}$$

Gibbs property

Gibbs property

- What is it?
- Where does it come from?
- How to think about it?

Gibbs property of KPZ line ensemble



$$\frac{d\mathbb{P}_{re}}{d\mathbb{P}_{BB}}(\mathcal{H}_2^{re}) = Z^{-1} W(\mathcal{H}_1, \mathcal{H}_2^{re}, \mathcal{H}_3)$$

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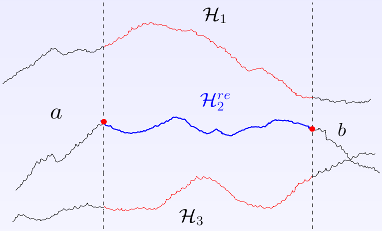

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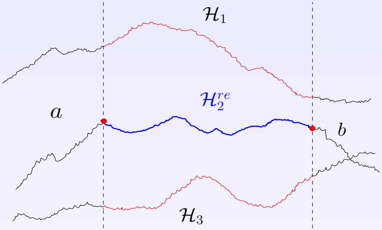

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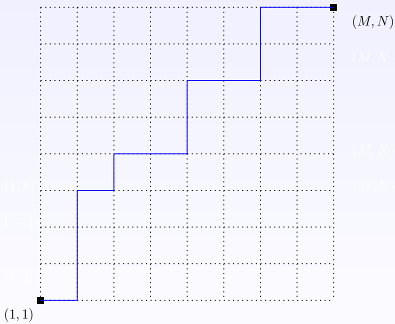
- Boundary determines the law of $\mathcal{H}$.
- Such a law = Brownian with interaction.

Log-Gamma directed polymers

$$X_{i,j} \sim \text{Log-Gamma},\ 1 \le i \le M, 1 \le j \le N$$



$$Z_1 := \sum_{\pi_1} \exp\left(\sum_{(i,j)\in\pi_1} X_{i,j}\right).$$

$$E_1 := Z_1$$

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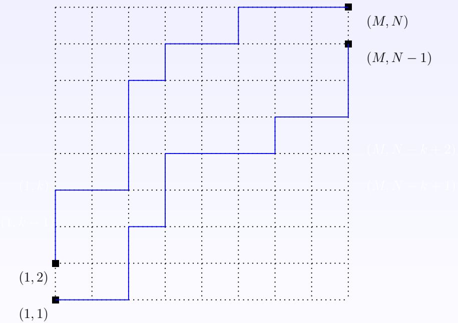

$$Z_2 := \sum_{(\pi_1,\pi_2)} \exp\left(\sum_{r=1}^{2} \sum_{(i,j)\in\pi_r} X_{i,j}\right)$$

$$E_2 := Z_2/Z_1$$

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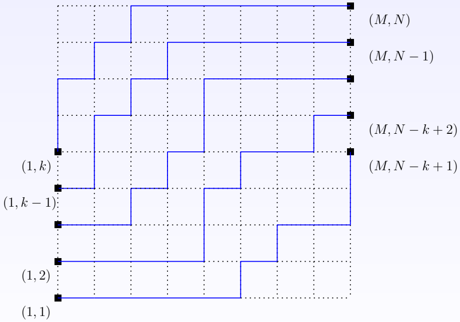

Π_k: k paths, up/right, non-intersecting

$$Z_k := \sum_{\pi\in\Pi_k} \exp\left(\sum_{r=1}^{k} \sum_{(i,j)\in\pi_r} X_{i,j}\right)$$

$$E_k := Z_k/Z_{k-1}$$

gRSK from the perspective of directed polymers

gRSK: bijection within $\mathbb{R}_{>0}^{M\times N}$.

Theorem (gRSK=DP)

For $Y \in \mathbb{R}_{>0}^{M\times N}$, $Y(M',N') = \{Y_{i,j} \,:\, 1 \le i \le M',\; 1 \le j \le N'\}$

$$\begin{aligned} t_{i-k+1,N-k+1} =& E_k(Y(i,N)),\; 1 \le i \le M, \\ t_{M-k+1,j-k+1} =& E_k(Y(M,j)),\; 1 \le j \le N. \end{aligned}$$

$y_{1,3}$			$y_{4,3}$
$y_{1,1}$			$y_{4,1}$

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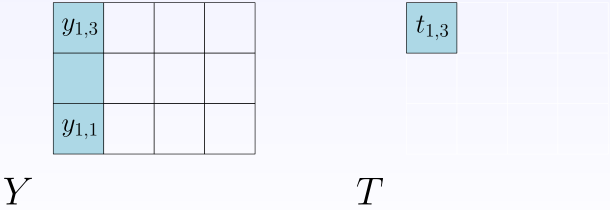

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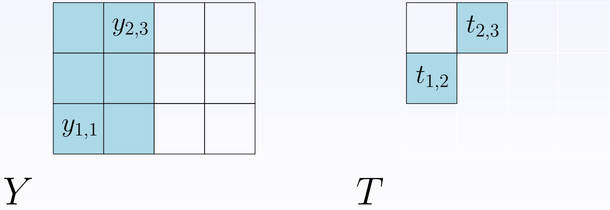

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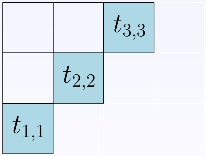

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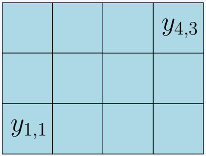

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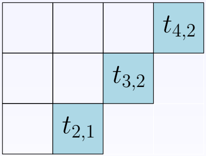

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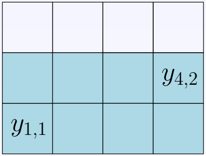

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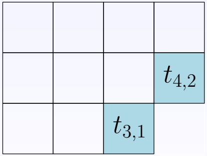

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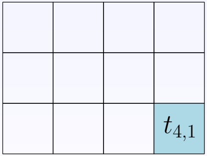
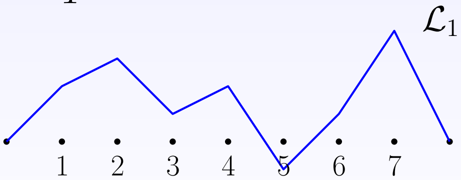


T

Embedding into a discrete line ensemble

$t_{1,3}$	$t_{2,3}$	$t_{3,3}$	$t_{4,3}$
			$t_{4,2}$
			$t_{4,1}$

T



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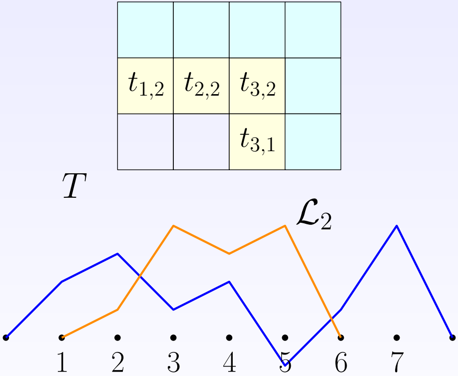

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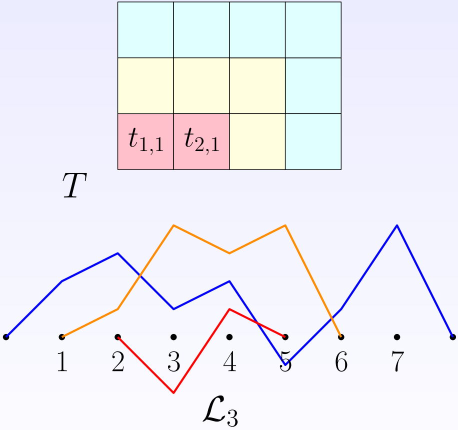
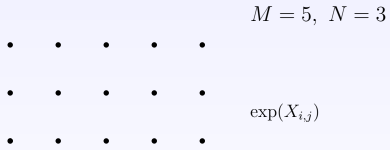


Gamma inputs

Gamma inputs: $\hat{\theta}_1, \cdots, \hat{\theta}_M$ and $\theta_1, \cdots, \theta_N$.

$$X = X_{i,j} \sim \text{Log-Gamma}(\hat{\theta}_i + \theta_j)$$



Gamma inputs

$$\mathrm{gRSK/DP} : Y \to T$$

$$\text{density} = \cdots \times \prod_{k=1}^{N} \exp\left(-\hat{\theta}_i(\mathcal{L}_k(i) - \mathcal{L}_k(i-1)) - e^{-(\mathcal{L}_k(i)-\mathcal{L}_k(i-1))}\right)$$
$$\times \prod_{k=1}^{N-1} \exp(-e^{\mathcal{L}_{k+1}(i)-\mathcal{L}_k(i)}) \times \cdots$$

blue: Log-Gamma random walk

red: interaction between different curves

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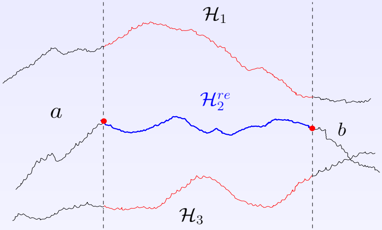
blue: Log-Gamma random walk $\longrightarrow$ Brownian motion

red: interaction between different curves $\longrightarrow$ interaction W

Gibbs property of KPZ line ensemble



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Application of Gibbs property

First stage of exploiting Gibbs property

e.g. $k = 2$, input: tail bound for $\mathfrak{h}_1$, stationary around $\frac{x^2}{2}$.

$$\mathbb{P}_{\mathfrak{h}} \left[\inf_{x \in [-1,1]} \left(\mathfrak{h}_2(x) + \frac{x^2}{2} \right) \leq -R \right] \leq e^{-R^{3/2}}$$

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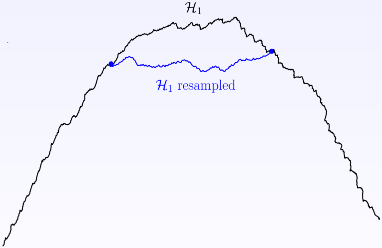

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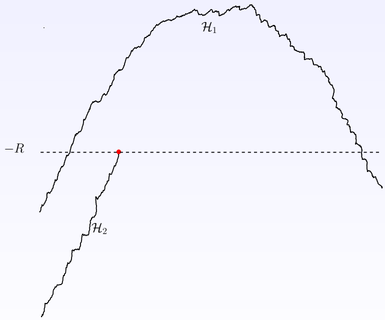

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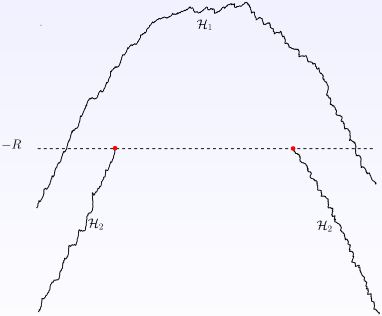
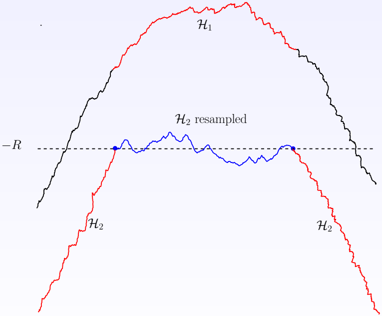


First stage of exploiting Gibbs property

e.g. $k = 2$, input: tail bound for $\mathfrak{h}_1$, stationary around $\frac{x^2}{2}$.

$$\mathbb{P}_{\mathfrak{h}}\left[\inf_{x\in[-1,1]}\left(\mathfrak{h}_2(x)+\frac{x^2}{2}\right)\leq -R\right]\leq e^{-R^{3/2}}$$



Summary

- KPZ universality class and KPZ line ensemble

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- The Gibbs property and its origin

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- The Gibbs property and its origin
- Application of Gibbs property