组合学 Combinatorics, 2022

Lecture 1 计数

* Let n be a positive integer.

  $[n] = \{1, 2, \cdots, n\}$.

* Let X be a set.

  $|X| = \#$ elements in X

* $n! = 1 \times 2 \times 3 \times \cdots \times n$ $\qquad 0! = 1$

  $(n)_k = n \times (n-1) \times \cdots \times (n-k+1)$.

§1. 二项式系数 Binomial coefficients

Definition Let X be a set of size n.

Let $2^X = \{A \subseteq X\}$

$\Rightarrow |2^X| = 2^n = 2^{|X|}$.

$2^{[3]} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Let $\binom{X}{k} = \{A \subseteq X : |A| = k\}$

Fact! $\binom{|X|}{k} = \left|\binom{X}{k}\right| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Fact 2.

(1.) $\binom{n}{k} = \binom{n}{n-k}$

(2.) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

pf: $\left|\binom{[n]}{k}\right| = \left|\binom{[n-1]}{k}\right| + \left|\binom{[n-1]}{k-1}\right|$.

组合方法证明 = Combinatorial proof.

Fact 3 The number of integer solutions to

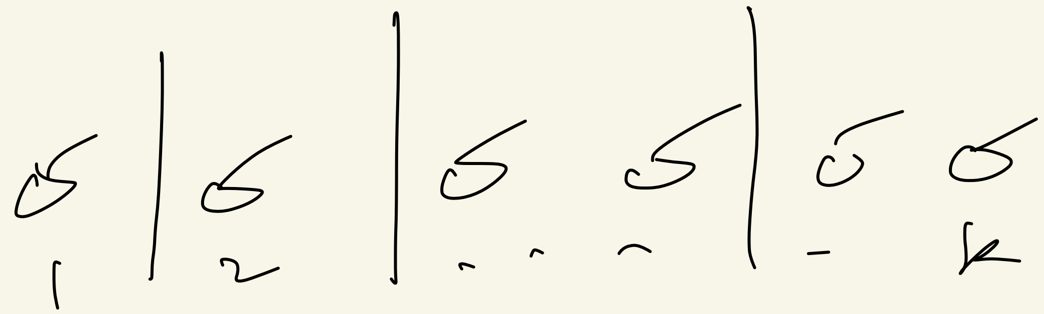
$$x_1 + x_2 + \cdots + x_n = k, \quad x_i \in \{0,1\}$$

$$= \binom{n}{k}$$

Fact 4. The problem, $x_i \geq 0$

$$= \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

k apples



$$\binom{n-1+k}{n-1}$$

Ex. 1. # Ways to select r of the integers $1, 2, \dots, n$ such that no two selected integers are consecutive.

§ 2. Counting functions

Def: Let X^Y be the set of all functions $f: Y \to X$.

$\Rightarrow |X^Y| = |X|^{|Y|}$ 单射

Fact 5. The number of injective functions $f: [r] \to [n]$ is $(n)_r$

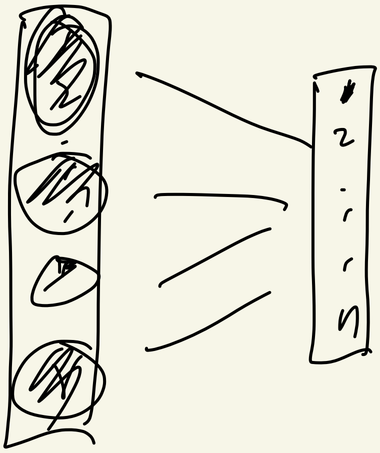
Def. (The Stirling number of the second kind)
Let $S(r, n)$ be the number of partitions of $[r]$ into n (non-ordered) non-empty sets.

Fact 6 The number of surjective functions

$f : [r] \to [n]$ is $S(r,n) \cdot n!$

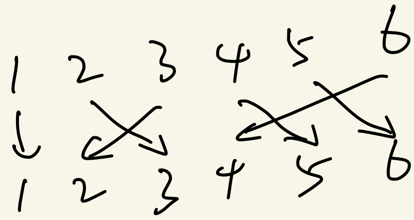
Pf:

$[r]$



Def: A permutation $f : X \to X$ is injective (双射)

例 $X = [n]$



$= 3$ cycles

Def (The Stirling number of the first kind)

Let $s(r,n)$ be the number of permutations of $[r]$ with exactly n cycles multiplied by $(-1)^{n-r}$.

§3. The Binomial Theorem (二项式定理)

Let $f(x)$ be a polynomial.

Let $[x^k]f$ be the coefficient of the term x^k in $f(x)$.

i.e.

$f(x) = -3 + 5x + 7x^3$

$[x^2]f = 0$

$[x^3]f = 7$

Fact 1. Let I_j be a set of non-negative integer for $j \in [n]$.

Let $f_j(x) = \sum_{i \in I_j} x^i$ for $j \in [n]$.

Let $\boxed{f(x) = \prod_{j=1}^{n} f_j(x)}$.

Then $[x^k]f = \sum_{\substack{i_1 + i_2 + \cdots + i_n = k \\ i_j \in I_j}} 1$.

Fact 2 Let $f_1, \ldots, f_n$ be polynomials and let $f = f_1 f_2 \cdots f_n$.

Then

$$[x^k]f = \sum_{\substack{i_1 + i_2 + \cdots + i_n = k \\ i_j \geq 0}} \left(\prod_{j=1}^{n} [x^{i_j}] f_j \right)$$

The Binomial Theorem For any real x and any integer $n>0$, $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$.

Pf. By Fact 1. $\boxed{}$

Vandermond's Convolution formula.

$$\binom{n+m}{k} = \sum_{\substack{i+j=k \\ i,j\geq 0}} \binom{n}{i}\binom{m}{j} \quad \checkmark$$

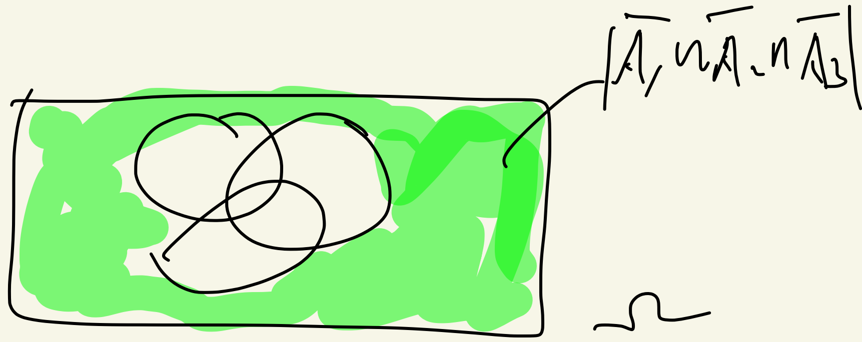
§4. Inclusion and exclusion principle (IEP)

Let Ω be a ground set and let $A_1, A_2, \ldots, A_n$ be subsets of Ω.

Write $A_i^c = \Omega \setminus A_i = \overline{A_i}$.

What is $|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$?

i.e $n=3$



Def. Let $A_\emptyset = \Omega$.

$A_{\{2,4,6\}} = A_2 \cap A_4 \cap A_6$

For $\emptyset \neq I \subseteq [n]$, let $A_I = \bigcap_{i \in I} A_i$.

For $k \geq 0$, $$S_k = \sum_{I \in \binom{[n]}{k}} |A_I|.$$

i.e. $S_0 = |\Omega|$,

$$S_1 = \sum_{i \in [n]} |A_i|$$

$$S_2 = \sum_{\{i,j\} \subseteq [n]} |A_i \cap A_j|$$

<u>Inclusion-exclusion Priciple / Formula</u>.

$$\left|\Omega \setminus \bigcup_{i=1}^{n} A_i\right| = \left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right|$$

$$= \sum_{k=0}^{n} (-1)^k S_k = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

Def. For $X \subseteq \Omega$, let its characterization function (<u>特征方程</u>) $$\mathbb{1}_X(x) = \begin{cases} 1, & x \in X \\ 0, & x \notin X \end{cases}$$

$(\mathbb{1}_X : \Omega \to \{0,1\})$

Pf 1. It suffices to show

$$\mathbb{1}_{\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}}(x) = \sum_{k=0}^{n}(-1)^k \sum_{I\in\binom{[n]}{k}} \mathbb{1}_{A_I}(x)$$

holds for any $x\in\Omega$.

Assume that x is contained in ℓ subsets, say $A_1, A_2, \dots, A_\ell$.

If $\ell=0 \iff x\in\overline{A_1}\cap\cdots\cap\overline{A_n}$

$$LHS=1, \quad RHS=1$$

Now $\ell\geq 1 \implies LHS=0$

$$RHS = 1-\binom{\ell}{1}+\binom{\ell}{2}-\binom{\ell}{3}+\cdots+(-1)^\ell\binom{\ell}{\ell}$$

$$=(1-1)^\ell = 0 \quad (\text{by the Binomial Thm})$$

$\square$

Pf 2. Let $A = A_1\cup A_2\cup\cdots\cup A_n$.

Key observation: $(\mathbb{1}_A-\mathbb{1}_{A_1})(\mathbb{1}_A-\mathbb{1}_{A_2})\cdots(\mathbb{1}_A-\mathbb{1}_{A_n})(x)$

$= 0 \quad \forall x \in \Omega.$

Expanding $\Rightarrow \ldots$ ∎

Example! Recall $S(n,k)$

(*) $S(n,k) = \#$ ways to partition $[n]$ into k non-empty sets (unordered)

(*) $S(n,k)\cdot k! = \#$ surjective functions $f: [n] \to [k]$

Prop! $S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$.

Pf: Let $\Omega = \{\text{all } f: [n] \to [k]\}$

Let $A_i = \{\text{all } f: [n] \to [k] \setminus \{i\}\}$ for $i \in [k]$

$\Rightarrow \{\text{all surjective functions}\} = \overline{A_1} \cap \overline{A_2} \cdots \cap \overline{A_k}$

By IEP,

$S(n,k)\cdot k! = |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}| = \sum_{i=0}^{k} (-1)^i S_i$

where $S_i = \sum_{I \in \binom{[k]}{i}} |A_I| = \binom{k}{i} (k-i)^n$

$f: [n] \to [k] \setminus I$ ∎

Example 2. Let $\phi(n) = \#$ integers $k \in [n]$ with $\gcd(k,n)=1$ 互素

最大公因子

Prop 2. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $p_1, \dots, p_r$ are primes. Then $\boxed{\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)}$

Pf: By IEP (练习). ∎

Prop 3. $\sum_{d|n} \phi(d) = n$ (练习)

Def (Möbius function)

$$\mu(d) = \begin{cases} 1, & \text{if } d = \text{product of an even number of distinct primes} \\ -1, & \text{if } d = \text{product of an odd number of distinct primes} \\ 0, & \text{otherwise.} \end{cases}$$

Prop 4. $\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{otherwise} \end{cases}$

Pf: If $n=1$, trivial ✓

Let $n = p_1^{a_1} \cdots p_r^{a_r} \geq 2$.

$$\sum_{d \mid n} \mu(d) = \sum_{\substack{i_1 \leq a_1 \\ i_r \leq a_r}} \mu(p_1^{i_1} \cdots p_r^{i_r}) \qquad i_j \in \{0,1\}$$

$$= \sum_{i=0}^{r} \binom{r}{i} (-1)^i = (1-1)^r = 0 \qquad \boxtimes$$

By prop 1 & 4,

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) \iff \frac{\phi(n)}{n} = \sum_{d \mid n} \frac{\mu(d)}{d}$$

Mobius inversion formula Let $f(n)$ and $g(n)$ be two functions defined for every positive integer n satisfying $f(n) = \sum_{d \mid n} g(d)$.

Then $g(n) = \sum_{d \mid n} \mu(d) f\left(\frac{n}{d}\right)$.

Pf. 练习 $\boxtimes$